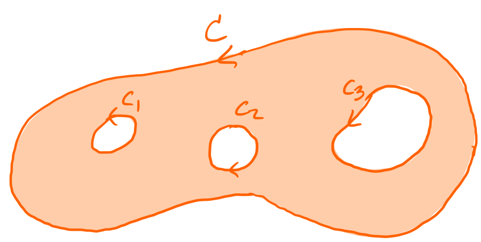
Exam 3 Solutions

1. Suppose $f(z) = u(x, y) + iv(x, y)$ is entire. Determine which statements are necessarily true.
   (a) $\int_C f(z)\,dz = 0$, where C is any closed contours C $\mathbb{C}$. TRUE
   (b) v is a harmonic conjugate of u. TRUE
   (c) f has an antiderivative on $\mathbb{C}$. TRUE
   (d) f is bounded. FALSE
   (e) $f'''(z)$ exists at all points $z \in \mathbb{C}$. TRUE
   (f) $|f(z)|$ has no minimum. FALSE
   (g) $u_{xx} - v_{xy} = 0$. TRUE

2. Suppose f is an entire function such that $f'(0) = 1 + i$. Show that f is not bounded.

   Solution: Since $f'(0) = 1 + i$, f is not a constant function. Thus by Liouville's Theorem, f cannot be bounded.

3. Consider the contours C, C_1, C_2, and C_3 depicted below.



   Fill in the blanks: If f is __analytic__ on the region enclosed by the contours and on the contours, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz \,\underline{-}\, \int_{C_2} f(z)\,dz \,\underline{+}\, \int_{C_3} f(z)\,dz$$

4. Let C be the negatively-oriented circle $|z| = 5$. Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z - \frac{\pi}{4}i)^n}\,dz$ for all integers $n \geq 0$. Explain your reasoning and simplify your answer (your answer will depend on n).

   Solution:

   If $n = 0$, then $\displaystyle\int_C \frac{e^{2z}}{(z - \frac{\pi}{4}i)^0}\,dz = \int_C e^{2z}\,dz = 0$, by the Cauchy-Goursat Theorem.

Let $n \geq 1$. Then since $f(z) = e^{2z}$ is analytic on and inside C and $\frac{\pi}{4}i$ is interior to C, which is **negatively-oriented**, by the Cauchy Integral Formula (for derivatives),

$$\int_C \frac{e^{2z}}{(z - \frac{\pi}{4}i)^n}\, dz = -\frac{2\pi i}{(n-1)!} f^{(n-1)}(\frac{\pi}{4}i) = -\frac{2\pi i 2^{n-1} e^{\frac{\pi}{2}i}}{(n-1)!} = \frac{2^n \pi}{(n-1)!}$$

5. Show that the complex series $\displaystyle\sum_{n=1}^{\infty} \frac{3i}{n^2 2^n}(z-i)^n$ converges on the disk $|z-i| \leq 2$ and diverges when $|z-i| > 2$? Explain your reasoning.

Solution:

First, $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{3i}{(n+1)^2 2^{n+1}}(z-i)^{n+1}}{\frac{3i}{n^2 2^n}(z-i)^n} \right| = \lim_{n\to\infty} \frac{n^2}{2(n+1)^2}|z-i| = \frac{1}{2}|z-i|.$

By the ratio test, the series converges if $\frac{1}{2}|z-i| < 1$, or $|z-i| < 2$, and diverges if $\frac{1}{2}|z-i| > 1$, or $|z-i| > 2$.

Now, when $|z-i| = 2$, $\displaystyle\sum_{n=1}^{\infty} \left| \frac{3i}{n^2 2^n}(z-i)^n \right| = \sum_{n=1}^{\infty} \frac{|3i|}{n^2 2^n}|z-i|^n = \sum_{n=1}^{\infty} \frac{3}{n^2}$, which converges by the $p-$series test. Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{3i}{n^2 2^n}(z-i)^n$ is absolutely convergent and thus convergent on $|z-i| = 2$.

Putting this all together, we have shown that the series converges on the disk $|z-i| \leq 2$ and diverges when $|z-i| > 2$.