Homework 1 Solutions

p. 4:

2.) a) Show that $\mathrm{Re}(iz) = -\mathrm{Im}\, z$

Let $z = x+iy$.

$\mathrm{Re}(iz) = \mathrm{Re}(i(x+iy)) = \mathrm{Re}(-y+ix) = -y$

$\mathrm{Im}(z) = \mathrm{Im}(x+iy) = y$

Thus $\mathrm{Re}(iz) = -y = -\mathrm{Im}(z)$

4.) Verify that $z = 1 \pm i$ satisfy $z^2 - 2z + 2 = 0$.

- $(1+i)^2 - 2(1+i) + 2 = (1+2i+i^2) - 2 - 2i + 2$
  $= 1 + 2i - 1 - 2 - 2i + 2 = 0$
- $(1-i)^2 - 2(1-i) + 2 = (1-2i+i^2) - 2 + 2i + 2$
  $= 1 - 2i - 1 - 2 + 2i + 2 = 0$

Thus $1 \pm i$ are solutions to $z^2 - 2z + 2 = 0$.
Alternatively, can use the quadratic formula.

p. 13:

2.) Show that $\mathrm{Re}\, z \le |\mathrm{Re}\, z| \le |z|$ and $\mathrm{Im}\, z \le |\mathrm{Im}\, z| \le |z|$

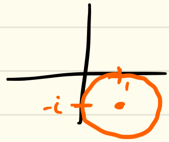
Let $z = x+iy$. Then

- $\mathrm{Re}\, z = x$, $|\mathrm{Re}\, z| = |x|$, and $|z| = \sqrt{x^2+y^2}$
  Thus $\mathrm{Re}\, z \le |\mathrm{Re}\, z|$ since $x \le |x|$
  and $|\mathrm{Re}\, z| \le |z|$ since $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$

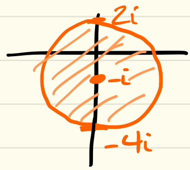
- $\mathrm{Im}\, z = y$, $|\mathrm{Im}\, z| = |y|$, and $|z| = \sqrt{x^2+y^2}$
  Thus, as above, $\mathrm{Im}\, z \le |\mathrm{Im}\, z| \le |z|$

5.) Sketch the set of points determined by the given conditions:
(a) $|z-1+i|=1$ (b) $|z+i|\leq 3$ (c) $|z-4i|\geq 4$

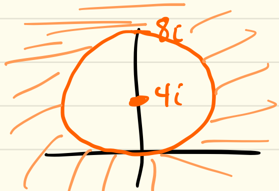
(a) $|z-1+i| = |z-(1-i)| = 1$ is a circle of radius 1 centered at $1-i$



(b) $|z+i|\leq 3$ is a disk of radius 3 centered at $-i$.



(c) $|z-4i|\geq 4$ is the exterior of the disk of radius 4 centered at $4i$



8.) Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$. Use algebra to show that $|(x_1+iy_1)(x_2+iy_2)| = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$ and deduce $|z_1z_2| = |z_1||z_2|$

$$|z_1z_2| = |(x_1+iy_1)(x_2+iy_2)| = |(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)|$$
$$= \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+y_1x_2)^2}$$
$$= \sqrt{x_1^2x_2^2+y_1^2y_2^2-2x_1x_2y_1y_2+x_1^2y_2^2+y_1^2x_2^2+2x_1x_2y_1y_2}$$
$$= \sqrt{x_1^2x_2^2+x_1^2y_2^2+y_1^2y_2^2+y_1^2x_2^2}$$
$$= \sqrt{x_1^2(x_2^2+y_2^2)+y_1^2(y_2^2+x_2^2)}$$
$$= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} = \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$
$$= |z_1||z_2|$$

9.) Use exercise 8 and induction to prove $|z^n| = |z|^n$ $(n = 1, 2, \dots)$

<u>Base Case</u>: $n=1$ $\quad |z^1| = |z| = |z|^1$

<u>Assume</u> $|z^m| = |z|^m$ where $m \geq 1$.

Then $|z^{m+1}| = |z^m \cdot z| = |z^m||z|$ by exercise 8

$= |z|^m |z|$ by the inductive assumption

$= |z|^{m+1}$

<u>p. 16</u>:

1.) Use the properties of conjugates and moduli to show: (a) $\overline{\bar{z} + 3i} = z - 3i$, (b) $\overline{iz} = -i\bar{z}$, (c) $\overline{(2+i)^2} = 3-4i$, and (d) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$.

(a) $\overline{\bar{z} + 3i} = \bar{\bar{z}} + \overline{3i} = z - 3i$

(b) $\overline{iz} = \bar{i}\,\bar{z} = -i\bar{z}$

(c) $\overline{(2+i)^2} = (\overline{2+i})^2 = (2-i)^2 = (2-i)(2-i) = 3-4i$

(d) $|(2\bar{z}+5)(\sqrt{2}-i)| = |2\bar{z}+5||\sqrt{2}-i| = |\overline{2z+5}|\sqrt{3} = \sqrt{3}|2z+5|$

10.)(a) Prove that z is real if and only if $\bar{z} = z$.

If z is real, then $z = x + i0 = x$ and so $\bar{z} = x = z$

If $z = x+iy$ and $z = \bar{z}$, then $x+iy = x-iy$

$\Rightarrow y = -y \Rightarrow y = 0 \Rightarrow z = x$ is real.

15.) a) Show that $|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2$

$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2$

$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2$

(b) Point out why $z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\,\mathrm{Re}(z_1\bar{z}_2) \le 2|z_1z_2|$

Since $\mathrm{Re}\, z = \frac{z+\bar{z}}{2}$ for all $z \in \mathbb{C}$,

$$z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\,\mathrm{Re}(z_1\bar{z}_2) \le 2|z_1\bar{z}_2| \text{ by problem 2 on page 13}$$
$$= 2|z_1||\bar{z}_2| \text{ by problem 8 on page 13}$$
$$= 2|z_1||z_2|$$

(c) Use (a) & (b) to obtain $|z_1+z_2|^2 \le (|z_1| + |z_2|)^2$ and note how the triangle inequality $|z_1+z_2| \le |z_1| + |z_2|$ follows.

$$|z_1+z_2|^2 = z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2$$
$$= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$
$$\le |z_1|^2 + |z_2|^2$$

Now, $(|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1z_2| \ge |z_1|^2 + |z_2|^2$

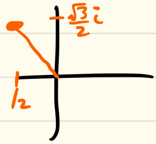
$$\Rightarrow \quad |z_1| + |z_2| \ge \sqrt{|z_1|^2 + |z_2|^2}$$

Thus $|z_1+z_2| = \sqrt{|z_1+z_2|^2} \le \sqrt{|z_1|^2 + |z_2|^2} \le |z_1| + |z_2|$

p.23

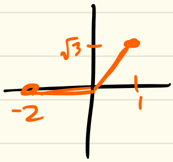
1.) Find the principal argument $\operatorname{Arg} z$ for
(a) $z = \dfrac{-2}{1+\sqrt{3}i}$ (b) $z = (\sqrt{3}-i)^6$

(a) $z = \dfrac{-2}{1+\sqrt{3}i} = \dfrac{-2}{1+\sqrt{3}i} \cdot \dfrac{1-\sqrt{3}i}{1-\sqrt{3}i} = \dfrac{-2+2\sqrt{3}i}{4} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$



Then $\tan\theta = \dfrac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$ has solution set in the 2nd quadrant $\arg z = \{\frac{2\pi}{3} + 2\pi k \mid k \in \mathbb{Z}\}$

Thus $\operatorname{Arg} z = \frac{2\pi}{3}$.



Alternatively, since $\arg(-2) = \{\pi + 2n\pi\}$ and $\arg(1+\sqrt{3}i) = \{\frac{\pi}{3} + 2n\pi\}$, we have $\arg\left(\dfrac{-2}{1+\sqrt{3}i}\right) = \arg(-2) - \arg(1+\sqrt{3}i)$

$= \{\frac{2\pi}{3} + 2n\pi\}$

$\Rightarrow \operatorname{Arg} z = \frac{2\pi}{3}$.

(b) First consider $\sqrt{3} - i$.

$\tan\theta = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$ has solution set in the 4th quadrant
$\arg(\sqrt{3}-i) = \{-\frac{\pi}{6} + 2\pi k \mid k \in \mathbb{Z}\}$

Thus $\arg(\sqrt{3}-i)^6 = 6\arg(\sqrt{3}-i) = \{-\pi + 2\pi k \mid k \in \mathbb{Z}\}$

and so $\operatorname{Arg}((\sqrt{3}-i)^6) = \pi$

2) Show that

(a) $|e^{i\theta}| = 1$, (b) $\overline{e^{i\theta}} = e^{-i\theta}$

a) $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

b) $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$
$= e^{-i\theta}.$

5.) By using exponential form, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$

(c) $(\sqrt{3}+i)^6 = -64$

a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = (e^{i\frac{\pi}{2}})(2e^{-i\frac{\pi}{3}})(2e^{i\frac{\pi}{6}}) = 4e^{i\frac{\pi}{3}}$
$= 4(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = 4(\frac{1}{2} + i\frac{\sqrt{3}}{2})$
$= 2(1+i\sqrt{3})$

c) $(\sqrt{3}+i)^6 = (2e^{i\frac{\pi}{6}})^6 = 2^6 e^{i\pi} = 64(-1) = -64$

10) Use de Moivre's formula to show

a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, (b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

By de Moivre, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

Now, $(\cos\theta + i\sin\theta)^3 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$
$= (\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta)(\cos\theta + i\sin\theta)$
$= \cos^3\theta + i\cos^2\theta\sin\theta + 2i\cos^2\theta\sin\theta + 2i^2\cos\theta\sin^2\theta$
$+ i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$
$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Since

$(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta$,

we have that:

a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

p. 30:

1.) Find the square roots of (a) $2i$ and (b) $1-\sqrt{3}i$ and express them in rectangular coordinates

a) $2i = 2e^{i\frac{\pi}{2}}$

$$(2i)^{1/2} = \left\{\sqrt{2}e^{i\frac{\pi}{4}}, \sqrt{2}e^{i\frac{5\pi}{4}}\right\} = \left\{\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right), \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\right\}$$

$$= \{1+i, -1-i\}$$

b) $1-\sqrt{3}i = 2e^{-i\frac{\pi}{3}}$

$$(1-\sqrt{3}i)^{1/2} = \left\{\sqrt{2}e^{-i\frac{\pi}{6}}, \sqrt{2}e^{i\frac{5\pi}{6}}\right\} = \left\{\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right), \sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right\}$$

$$= \left\{\frac{\sqrt{3}-i}{\sqrt{2}}, -\frac{\sqrt{3}-i}{\sqrt{2}}\right\}$$

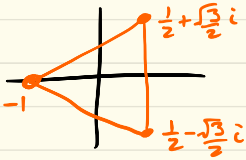
4.) In each case, find all roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root.

(a) $(-1)^{1/3}$, (b) $8^{1/6}$

a) $-1 = e^{i\pi}$

$$(-1)^{1/3} = \left\{e^{i\frac{\pi}{3}}, e^{i\pi}, e^{i\frac{5\pi}{3}}\right\} = \left\{\frac{1}{2}+\frac{\sqrt{3}}{2}i, -1, \frac{1}{2}-\frac{\sqrt{3}}{2}i\right\}$$

The principal root is $e^{\frac{i\pi}{3}}$



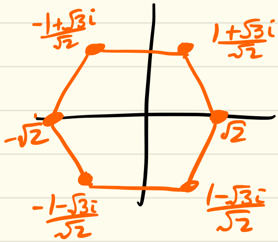
b) $8 = 8e^{i0}$

$8^{1/6} = \{8^{1/6}e^{i0},\ 8^{1/6}e^{i\frac{2\pi}{6}},\ 8^{1/6}e^{i\frac{4\pi}{6}},\ 8^{1/6}e^{i\frac{6\pi}{6}},\ 8^{1/6}e^{i\frac{8\pi}{6}},\ 8^{1/6}e^{i\frac{10\pi}{6}}\}$

$= \{\sqrt{2}, \sqrt{2}(\frac{1}{2}+\frac{\sqrt{3}}{2}i), \sqrt{2}(-\frac{1}{2}+\frac{\sqrt{3}}{2}i), -\sqrt{2}, \sqrt{2}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i), \sqrt{2}(\frac{1}{2}-\frac{\sqrt{3}}{2}i)\}$

$= \{\pm\sqrt{2},\ \pm\frac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\frac{1-\sqrt{3}i}{\sqrt{2}}\}$

The principal root is $\sqrt{2}$



8.) (a) Prove that $az^2+bz+c=0$ $(a\neq 0)$ has solution $z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$, where $a,b,c \in \mathbb{C}$ and both square roots are to be considered when $b^2-4ac \neq 0$, by completing the square.

(b) Find the roots of $z^2+2z+(1-i)=0$.

(a) $az^2+bz+c=0$

$$z^2+\frac{b}{a}z = -\frac{c}{a}$$

$$z^2+\frac{b}{a}z+\left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\left(z+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \Rightarrow z+\frac{b}{2a} = \left(\frac{b^2-4ac}{4a^2}\right)^{1/2} = \frac{(b^2-4ac)^{1/2}}{2a}$$

$$\Rightarrow z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$$

(b) $z = \frac{-2+(4-4(1)(1-i))^{1/2}}{2(1)} = \frac{-2+(4i)^{1/2}}{2} = -1+i^{1/2}$

Now, $i^{1/2} = \{e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}}\} = \{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\}$

Thus, $z = -1+i^{1/2} = \{(\frac{\sqrt{2}}{2}-1)+\frac{\sqrt{2}}{2}i, (-\frac{\sqrt{2}}{2}-1)-\frac{\sqrt{2}}{2}i\}$.