Homework 10 Solutions

1.) a) $\frac{4}{3-z} = \frac{4}{2-(z-1)} = 2\frac{1}{1-\frac{z-1}{2}} = 2\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n-1}}(z-1)^n,$

where $\left|\frac{z-1}{2}\right|<1 \Rightarrow |z-1|<2$

b) $\cos(iz+1) = \sum_{n=0}^{\infty}\frac{(-1)^n(iz+1)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n i^{2n}(z-i)^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(2n)!}(z-i)^{2n} = \sum_{n=0}^{\infty}\frac{(z-i)^{2n}}{(2n)!},$

where $|iz+1|<\infty \Rightarrow |z-i|<\infty$

2.) a) $f(z)=\tan z = \frac{\sin z}{\cos z}$ is analytic everywhere except when $\cos z=0$, which is when $z=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\ldots$ Thus f is analytic in the disk $|z|<\frac{\pi}{2}$ and by Taylor's thm, the Maclaurin series converges to $\tan z$ at all points in $|z|<\frac{\pi}{2}$.
$\Rightarrow |z|=\frac{\pi}{2}$ is the largest circle within which the series converges to $\tan z$.

b) $f(0)=0$
$f'(0)=\sec^2(0)=1$
$f''(0)=2\sec^2(0)\tan(0)=0$
$f'''(0)=4\sec^2(0)\tan^2(0)+2\sec^4(0)=2$

Thus the first 2 nonzero terms are $\frac{f'(0)}{1!}z=z$ and $\frac{f^{(3)}(0)}{3!}z^3=\frac{2}{3!}z^3=\frac{1}{3}z^3.$

3) The Maclaurin series for ze^{z^2} is $ze^{z^2} = z\left(\sum_{n=0}^{\infty} \frac{(z^2)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$.

The general formula is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$, where $f(z) = ze^{z^2}$

Since $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$ has no even power terms,

The $(2n)^{th}$ term of the series is $\frac{f^{(2n)}(0)}{(2n)!} z^{2n} = 0 \cdot z^{2n}$

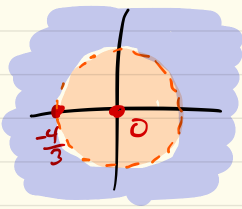
$\Rightarrow f^{(2n)}(0) = 0$ for all $n \geq 0$

The $(2n+1)^{th}$ term of the series is $\frac{f^{(2n+1)}(0)}{(2n+1)!} z^{2n+1} = \frac{1}{n!} z^{2n+1}$

$\Rightarrow f^{(2n+1)}(0) = \frac{(2n+1)!}{n!} = (2n+1)(2n)(2n-1)\cdots(n+1)$

4) a) $\frac{1}{z^2}\cosh z = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n-2}}{(2n)!}$, $0 < |z| < \infty$

b) $\frac{4}{3z^3+4z^2}$ is analytic everywhere except when $3z^3+4z^2 = z^2(3z+4) = 0 \Leftrightarrow z = -\frac{4}{3}, 0$



Thus there are Laurent series expansions on $0 < |z| < \frac{4}{3}$ and on $|z| > \frac{4}{3}$

We must consider each region separately.

• When $0<|z|<\frac{4}{3}$:

$$\frac{4}{3z^3+4z^2} = \frac{4}{z^2}\,\frac{1}{3z-4} = \frac{-1}{z^2}\,\frac{1}{1-(-\frac{3}{4}z)}$$

$$= -\frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{3}{4}z\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}3^n}{4^n}z^{n-2}$$

• When $|z|>\frac{4}{3}$, then $\left|\frac{1}{z}\right|<\frac{3}{4} \Rightarrow \left|\frac{4}{3z}\right|<1$

$$\Rightarrow \frac{4}{3z^3+4z^2} = \frac{4}{z^3}\,\frac{1}{3+\frac{4}{z}} = \frac{4}{3z^3}\,\frac{1}{1-(-\frac{4}{3z})} = \frac{4}{3z^3}\sum_{n=0}^{\infty}\left(\frac{-4}{3z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 4^{n+1}}{3^{n+1}z^{n+3}}$$

c) $$\sin\left(\frac{1}{z^3}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{z^3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,z^{6n+3}}, \quad 0<|z|<\infty$$

5.) a) $$\sum_{n=0}^{\infty}\frac{z^{2n-2}}{(2n)!} = \frac{1}{z^2}+\frac{1}{2}+\frac{z^2}{4!}+\frac{z^4}{6!}+\cdots$$

Since the largest negative exponent is 2, $z=0$ is a pole of order 2.

b) On the deleted neighborhood $0<|z|<\frac{4}{3}$,

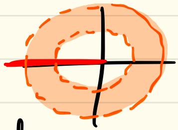
$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}3^n}{4^n}z^{n-2} = -\frac{1}{z^2}+\frac{3}{4}\frac{1}{z}-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3z+\cdots$$

$\Rightarrow$ $z=0$ is a pole of order 2.

c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,z^{6n+3}}$ has infinitely many terms with negative exponents

$\Rightarrow$ $z=0$ is an essential singularity.

6.) a) Since every point on the negative real axis is a singularity of $\operatorname{Log} z$, $\operatorname{Log} z$ is not analytic on any annular neighborhood of 0. So, Laurent's theorem doesn't apply.



b) No, for the same reason as for part (a).

c) Yes. If $z_0 \neq 0$ is any point not on the negative real axis, then $\operatorname{Log} z$ is analytic at z_0. Thus, by definition, $\operatorname{Log} z$ is analytic on an ε-neighborhood of z_0, $|z - z_0| < \varepsilon$. By Taylor's thm, $\operatorname{Log} z$ has a Taylor series representation for all z in $|z - z_0| < \varepsilon$.