Homework 11 Solutions

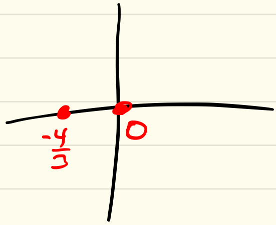
1.) a) $f(z) = \frac{1}{z}\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n)!} = \frac{1}{z} + \frac{1}{2!}z + \frac{1}{4!}z^3 + \cdots, \quad 0<|z|<\infty$

Since the coefficient in front of $\frac{1}{z}$ is 1, $\operatorname*{Res}_{z=0} f(z) = 1$

Since f is analytic on & inside the unit circle C except at $z=0$, $\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i$.

b) $g(z) = \frac{4}{3z^3+4z^2}$ has 2 isolated singularities $z=0$ and $z=-\frac{4}{3}$

On the last homework, we built the Laurent series centered at 0:



$$\frac{4}{3z^3+4z^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^n}{4^n} z^{n-2} \quad \text{on } 0<|z|<\frac{4}{3}$$

$$= -\frac{1}{z^2} + \frac{3}{4}\frac{1}{z} - \frac{3^2}{4^2} + \frac{3^3}{4^3}z - \cdots \Rightarrow \operatorname*{Res}_{z=0} g(z) = \frac{3}{4}$$

Since g is analytic on & inside the unit circle C except at $z=0$, $\int_C g(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} g(z) = \frac{3\pi i}{2}$.

c) $h(z) = \sin\left(\frac{1}{z^3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,z^{6n+3}} = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z^9} + \cdots, \quad 0<|z|<\infty$

Since there is no $\frac{1}{z}$ term, its coefficient is 0

$\Rightarrow \operatorname*{Res}_{z=0} h(z) = 0$

Thus $\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} h(z) = 0$.

2) $f(z) = \dfrac{z+1}{z^2-2z}$

(a) Since $z^2-2z=0 \Leftrightarrow z=0,2$,
f has 2 isolated singular points $z=0,2$ enclosed by C

Laurent Expansion at $z=0$:

$$\frac{z+1}{z^2-2z} = -\frac{z+1}{z}\;\frac{1}{2-z} = -\frac{z+1}{2z}\;\frac{1}{1-\frac{1}{2}z}$$

$$= \left(-\frac{1}{2}-\frac{1}{2z}\right)\sum_{n=0}^{\infty}\left(\tfrac{1}{2}z\right)^n, \quad 0<|z|<\frac{1}{2}$$

$$= \left(-\frac{1}{2}-\frac{1}{2z}\right)\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$= -\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}+ - - - - -$$

$$-\frac{1}{2}\frac{1}{z}-\frac{z}{4} - - - - - -$$

$$\Rightarrow \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}$$

Laurent Expansion at $z=2$:

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z-2}\cdot\frac{1}{z} = \frac{z+1}{z-2}\cdot\frac{1}{2+(z-2)}$$

$$= \frac{(z-2)+3}{2(z-2)}\;\frac{1}{1+\frac{z-2}{2}} = \left(\frac{1}{2}+\frac{3}{2(z-2)}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n$$

$$= \left(\frac{1}{2}+\frac{3}{2(z-2)}\right)\left(1-\frac{(z-2)}{2}+\frac{(z-2)^2}{4} - - - - - \right) =$$

$$= \frac{1}{2} - \frac{(z-2)}{4} + \frac{(z-2)^2}{8} - - - -$$

$$+ \frac{3}{2}\frac{1}{z-2} - \frac{3}{4} + \frac{3(z-2)}{8} - - - - -$$

$$\Rightarrow \operatorname*{Res}_{z=2} f(z) = \frac{3}{2}$$

b) By the Cauchy Residue Theorem,

$$\int_C f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=2} f(z)\right] = 2\pi i\left[-\frac{1}{2}+\frac{3}{2}\right] = 2\pi i$$

c) $\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$

$$g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\frac{1}{z}+1}{\left(\frac{1}{z}\right)^2 - \frac{2}{z}}\right) = \frac{\frac{1}{z}+1}{1-2z} = \frac{z+1}{z(1-2z)}$$

<u>Laurent Series of g at 0:</u>

$$\frac{z+1}{z(1-2z)} = \frac{z+1}{z}\frac{1}{1-2z} = \left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}(2z)^n$$

$$= \left(1+\frac{1}{z}\right)\left(1+2z+4z^2+8z^3+\cdots\right)$$

$$= 1+2z+4z^2+ - - - - + \frac{1}{z}+2+2z^2+ - - -$$

$$\Rightarrow \operatorname*{Res}_{z=0} g(z) = 1$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} g(z) = -1$$

d) $\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=\infty} f(z) = -\frac{1}{2} + \frac{3}{2} - 1 = 0$

e) Let $z_1, \ldots, z_K$ be the isolated singularities of f
Let C be a positively oriented simple closed contour enclosing $z_1, \ldots, z_K$.
Then by the Cauchy Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{K} \operatorname*{Res}_{z=z_i} f(z)$$

and by definition, $\int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$

Thus $-2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i \sum_{i=1}^{K} \operatorname*{Res}_{z=z_i} f(z)$

$$\Rightarrow \quad 2\pi i \sum_{i=1}^{K} \operatorname*{Res}_{z=z_i} f(z) + 2\pi i \operatorname*{Res}_{z=\infty} f(z) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{K} \operatorname*{Res}_{z=z_i} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

3) (a) $f(z) = \frac{\sin z}{(2z-\pi)^3} = \frac{\frac{\sin z}{8}}{(z-\frac{\pi}{2})^3}$

Let $\phi(z) = \frac{\sin z}{8}$. Then ϕ is analytic and nonzero at $z = \frac{\pi}{2}$

Thus $f(z) = \frac{\sin z}{(2z-\pi)^3}$ has a pole of order 3 at $z = \frac{\pi}{2}$

and $\operatorname*{Res}_{z=-9} f(z) = \frac{\phi''(\frac{\pi}{2})}{2!} = \frac{\frac{-\sin(\frac{\pi}{2})}{8}}{2} = -\frac{1}{16}$

b) $g(z) = \dfrac{z+3}{z^2(z^2+2)} = \dfrac{z+3}{z(z-\sqrt{2}i)(z+\sqrt{2}i)}$

has 3 isolated singular points: $z=0$, $z=\sqrt{2}i$, $z=-\sqrt{2}i$

Let $\phi_1(z) = \dfrac{z+3}{z^2(z-\sqrt{2}i)}$, $\phi_2(z) = \dfrac{z+3}{z^2(z+\sqrt{2}i)}$, $\phi_3(z) = \dfrac{z+3}{z^2+2}$

- $\phi_1(z)$ is analytic & nonzero at $-\sqrt{2}i$

  $\Rightarrow z=-\sqrt{2}i$ is a pole of order 1 and

  $$\operatorname*{Res}_{z=-\sqrt{2}i} g(z) = \phi_1(-\sqrt{2}i) = \frac{1}{4} + \frac{3}{4\sqrt{2}}i$$

- $\phi_2(z)$ is analytic & nonzero at $z=\sqrt{2}i$

  $\Rightarrow z=\sqrt{2}i$ is a pole of order 1 and

  $$\operatorname*{Res}_{z=\sqrt{2}i} g(z) = \phi_2(\sqrt{2}i) = -\frac{1}{4} - \frac{3}{4\sqrt{2}}i$$

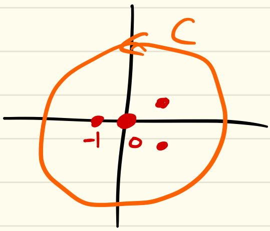
- $\phi_3(z)$ is analytic & nonzero at $z=0$

  $\Rightarrow z=0$ is a pole of order 2

  and $$\operatorname*{Res}_{z=0} g(z) = \phi_3'(0) = \left.\frac{(z^2+2)-2z(z+3)}{(z^2+2)^2}\right|_{z=0} = \frac{1}{2}.$$

4.) $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$ has four isolated singular points enclosed by C: $z=0$, and the 3rd roots of -1.

Moreover, f is analytic on the region $|z| > 2$.
Thus we can use the residue at infinity to calculate the contour integral.



Now, $\displaystyle\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$

$$g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\left(\frac{1}{z}\right)^3 e^{1/(1/z)}}{1+(1/z)^3} = \frac{e^z}{z^2(z^3+1)}$$

Notice, $z=0$ is a pole of order 2 of g since $g(z) = \dfrac{e^z/z^3+1}{z^2}$ and $\phi(z) = \dfrac{e^z}{z^3+1}$ is analytic and nonzero at $z=0$.

Thus $\displaystyle\operatorname*{Res}_{z=0} g(z) = \frac{\phi'(0)}{1!} = \frac{e^0(0^3+1) - 3(0)^2 e^0}{(0^3+1)^2} = 1$

Thus $\displaystyle\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} g(z) = -1$

and so $\displaystyle\int_C f(z) = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i.$

5.) We can calculate $\int_C \frac{P(z)}{Q(z)} dz$ by calculating the residue of $\frac{P(z)}{Q(z)}$ at ∞.

Let $g(z) = \frac{1}{z^2} \frac{P(\frac{1}{z})}{Q(\frac{1}{z})}$.

Then $g(z) = \frac{1}{z^2} \frac{a_n(\frac{1}{z})^n + \cdots + a_1 \frac{1}{z} + a_0}{b_m(\frac{1}{z})^m + \cdots + b_1 \frac{1}{z} + b_0} \frac{z^m}{z^m}$

$$= \frac{1}{z^2} \frac{a_n z^{m-n} + \cdots + a_1 z^{m-1} + a_0 z^m}{b_m + \cdots + b_1 z^{m-1} + b_0 z^m}$$

$$= \frac{1}{z^2} \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_n z^{m-n}}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$$

$$= \frac{a_0 z^{m-2} + a_1 z^{m-3} + \cdots + a_n z^{m-n-2}}{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}$$

Since $m \geq n+2$, $m-n-2 \geq 0$ and so the numerator of $g(z)$ is a polynomial and thus it is analytic everywhere.

Moreover, since $b_m \neq 0$, the denominator does not equal 0 at $z=0$.

Thus $\frac{a_0 z^{m-2} + \cdots + a_n z^{m-n-2}}{b_0 z^m + \cdots + b_m}$ is analytic at $z=0$

$\Rightarrow$ the Laurent series centered at $z=0$ has no negative exponent terms

$\Rightarrow \operatorname{Res}_{z=0} g(z) = 0 \Rightarrow \int_C \frac{P(z)}{Q(z)} dz = -2\pi i \operatorname{Res}_{z=\infty} f(z) = 2\pi i \operatorname{Res}_{z=0} g(z) = 0$