Homework 2 Solutions

p. 43

1) Describe the domain of (c) $f(z)=\frac{z}{z+\bar{z}}$ and (d) $\frac{1}{1-|z|^2}$

(c) $f(z)=\frac{z}{z+\bar{z}}=\frac{x+iy}{x+iy+x-iy}=\frac{x+iy}{2x} \Rightarrow \text{Domain}=\{z\in\mathbb{C} \mid \operatorname{Re} z\neq 0\}$

(d) $f(z)=\frac{1}{1-|z|^2} \Longrightarrow \text{Domain}=\{z\in\mathbb{C} \mid |z|\neq 1\}$

4.) Write $f(z)=z+\frac{1}{z}$ $(z\neq 0)$ in the form $f(z)=u(r,\theta)+i\,v(r,\theta)$

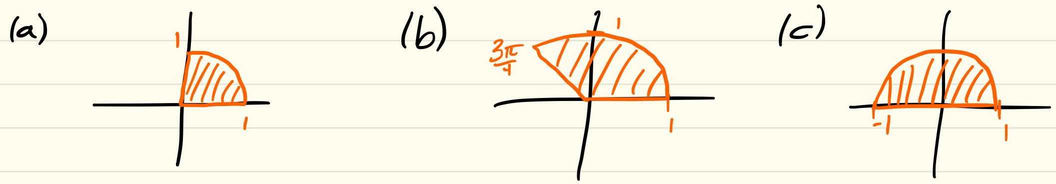
$$f(z)=z+\frac{1}{z}=x+iy+\frac{1}{x+iy}=x+iy+\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i$$

$$=r\cos\theta+ir\sin\theta+\frac{r\cos\theta}{r^2}-i\frac{r\sin\theta}{r^2}$$

$$=\left(r\cos\theta+\frac{\cos\theta}{r}\right)+i\left(r\sin\theta-\frac{\sin\theta}{r}\right)$$

8.) Sketch the region onto which the sector $r\leq 1$, $0\leq\theta\leq\frac{\pi}{4}$ is mapped by the transformations

(a) $w=z^2$ (b) $w=z^3$ (c) $w=z^4$



P. 89:

1.) Show that (a) $e^{2\pm 3\pi i} = -e^2$, (b) $e^{\frac{2+\pi i}{4}} = \sqrt{\frac{e}{2}}(1+i)$, (c) $e^{z+\pi i} = -e^z$

(a) $e^{2\pm 3\pi i} = e^2 e^{\pm 3\pi i} = e^2(\cos(\pm 3\pi) + i\sin(\pm 3\pi)) = e^2(-1) = -e^2$

(b) $e^{\frac{2+\pi i}{4}} = e^{1/2} e^{\frac{i\pi}{4}} = e^{1/2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) = e^{1/2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})$

$= \sqrt{\frac{e}{2}}(1+i)$

(c) $e^{z+\pi i} = e^z e^{\pi i} = e^z(\cos(\pi) + i\sin(\pi)) = -e^z$.

6.) Show that $|e^{z^2}| \leq e^{|z|^2}$

$$|e^{z^2}| = |e^{(x+iy)^2}| = |e^{(x^2-y^2)+2xyi}| = |e^{x^2-y^2}||e^{2xyi}|$$
$$= e^{x^2-y^2}|e^{2xyi}| = e^{x^2-y^2}|\cos(2xy) + i\sin(2xy)|$$
$$= e^{x^2-y^2}(\cos^2(2xy) + \sin^2(2xy))$$
$$= e^{x^2-y^2} \leq e^{x^2+y^2} = e^{|z|^2}.$$

10.) (a) Show that if e^z is real, then $\operatorname{Im} z = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$

Let $z = x+iy$ and assume $e^z = e^x e^{iy}$ is real.

Then $e^{iy} = \cos y + i\sin y$ is real

$\Rightarrow \sin y = 0 \Rightarrow \operatorname{Im} z = y = n\pi, \; n \in \mathbb{Z}$.

p. 95:

1) Show that (a) $\mathrm{Log}(-ei) = 1 - \frac{\pi}{2} i$, (b) $\mathrm{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$

(a) $-ei = e(-i) = e e^{-i\frac{\pi}{2}}$

So $\mathrm{Log}(-ei) = \ln e + i\left(-\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} i$

(b) $1 - i = \sqrt{2} e^{-i\frac{\pi}{4}}$

So $\mathrm{Log}(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4}\right) = \ln\sqrt{2} - \frac{\pi}{4} i$

2.) Show that (a) $\log e = 1 + 2n\pi i$, (b) $\log i = \left(2n + \frac{1}{2}\right)\pi i$

(c) $\log(-1+\sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$

(a) $e = e e^{i0}$

So $\log e = \ln e + (0 + 2\pi n) i = 1 + 2\pi n i$, $n \in \mathbb{Z}$

(b) $i = e^{i\frac{\pi}{2}}$

So $\log i = \ln(1) + \left(\frac{\pi}{2} + 2\pi n\right) i = \left(2n + \frac{1}{2}\right)\pi i$, $n \in \mathbb{Z}$

(c) $-1 + \sqrt{3} i = 2 e^{i\frac{2\pi}{3}}$

So $\log(-1+\sqrt{3}i) = \ln 2 + \left(\frac{2\pi}{3} + 2\pi n\right) i$, $n \in \mathbb{Z}$.

3.) Show that $\mathrm{Log}(i^3) \neq 3\,\mathrm{Log}(i)$

$i^3 = \left(e^{i\frac{\pi}{2}}\right)^3 = e^{i\frac{3\pi}{2}} = e^{-i\frac{\pi}{2}} \Rightarrow \mathrm{Log}(i^3) = \ln(1) + i\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} i$

On the other hand, $3\,\mathrm{Log}(i) = 3\left(\ln(1) + i\frac{\pi}{2}\right) = \frac{3\pi}{2} i$

Thus $\mathrm{Log}(i^3) \neq 3\,\mathrm{Log}(i)$.

5.) (a) Show $i^{1/2} = \{e^{i\frac{\pi}{4}}, e^{\frac{i5\pi}{4}}\}$. Then show $\log(e^{i\frac{\pi}{4}}) = (2n+\frac{1}{4})\pi i$
and $\log(e^{i\frac{5\pi}{4}}) = [(2n+1)+\frac{1}{4}]\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$
conclude that $\log(i^{1/2}) = (n+\frac{1}{4})\pi i$

(b) Show that $\log(i^{1/2}) = \frac{1}{2}\log(i)$

(a) $i = e^{i\frac{\pi}{2}}$. Thus $i^{1/2} = \{e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}}\}$

Now, $\log(e^{i\frac{\pi}{4}}) = \ln(1) + (\frac{\pi}{4} + 2\pi n)i = (2n+\frac{1}{4})\pi i, \quad n \in \mathbb{Z}$

$\log(e^{i\frac{5\pi}{4}}) = \ln(1) + (\frac{5\pi}{4} + 2\pi n)i = (2n+\frac{5}{4})\pi i, \quad n \in \mathbb{Z}$

$= ((2n+1)+\frac{1}{4})\pi i, \quad n \in \mathbb{Z}$

Thus $\log(i^{1/2}) = \{(2n+\frac{1}{4})\pi i, ((2n+1)+\frac{1}{4})\pi i\}$

$= (m+\frac{1}{4})\pi i, \quad m \in \mathbb{Z}$.

(b) $\frac{1}{2}\log(i) = \frac{1}{2}\log(e^{i\frac{\pi}{2}}) = \frac{1}{2}(\ln(1) + i(\frac{\pi}{2}+2\pi n)) = (n+\frac{1}{4})\pi i, n \in \mathbb{Z}$

Thus $\log(i^{1/2}) = \frac{1}{2}\log(i)$.

8.) Find all roots of $\log z = i\frac{\pi}{2}$

Since $\log z = \ln|z| + i(\operatorname{Arg} z + 2\pi n)$, $n \in \mathbb{Z}$
we have $\ln|z| + i(\operatorname{Arg} z + 2\pi n) = \frac{i\pi}{2}$
Thus $\ln|z| = 0 \Rightarrow |z| = 1$ and $\operatorname{Arg} z + 2\pi n = \frac{\pi}{2}$
$\Rightarrow n = 0, \operatorname{Arg} z = \frac{\pi}{2}$

Thus $z = i$.

p.103:

1.) (a) Show that $(1+i)^i = e^{(-\frac{\pi}{4}+2n\pi)} e^{(i\frac{\ln 2}{2})}$

$$(1+i)^i = e^{i\log(1+i)} = e^{i(\ln\sqrt{2}+i(\frac{\pi}{4}+2\pi m))} = e^{(-\frac{\pi}{4}+2\pi n)+i\ln\sqrt{2}}$$
$$= e^{(-\frac{\pi}{4}+2\pi n)} e^{i\frac{\ln 2}{2}}, \quad n=-m \in \mathbb{Z}.$$

5.) Show that the principal n^{th} root of z_0 is the same as the principal value of $z_0^{1/n}$.

Let $z_0 = r_0 e^{i\Theta_0}$, where $-\pi < \Theta_0 \leq \pi$. The principal n^{th} root is $z_0^{1/2} = \sqrt[n]{r_0}\, e^{i\frac{\Theta_0}{n}}$. The principal value of $z_0^{1/n}$ is $e^{\frac{1}{n}\operatorname{Log} z_0} = e^{\frac{1}{n}(\ln r_0 + i\Theta_0)} = e^{\ln(\sqrt[n]{r_0}) + i\frac{\Theta_0}{n}} = e^{\ln(\sqrt[n]{r_0})} e^{i\frac{\Theta_0}{n}} = \sqrt[n]{r_0}\, e^{i\frac{\Theta_0}{n}}$

p.111

2(a) Prove that $\sinh 2z = 2\sinh z\cosh z$ by using definitions of $\sinh z$ and $\cosh z$.

$$2\sinh z\cosh z = 2\left(\frac{e^z - e^{-z}}{2}\right)\left(\frac{e^z + e^{-z}}{2}\right)$$
$$= \frac{e^{2z} - e^{-2z}}{2} = \sinh 2z$$

7.) (a) Show that $\sinh(z+\pi i) = -\sinh z$

$$\sinh z(z+\pi i) = \frac{e^{z+\pi i} - e^{-z-\pi i}}{2} = \frac{e^z e^{\pi i} - e^{-z}e^{-\pi i}}{2}$$
$$= \frac{-e^z + e^{-z}}{2} = -\left(\frac{e^z - e^{-z}}{2}\right) = -\sinh z$$

Additional Problem

1.) (a) ℓ passes through $(0,0,1)$ and is parallel to the vector $\vec{NP} = \langle a,b,c\rangle - \langle 0,0,1\rangle = \langle a,b,c-1\rangle$

Thus ℓ has equation

$$\vec{r}(t) = \langle 0,0,1\rangle + t\langle a,b,c-1\rangle = \langle at, bt, (c-1)t+1\rangle$$

(b) ℓ intersects the xy-plane when its z-component is 0, or $(c-1)t+1=0 \Rightarrow t = -\frac{1}{c-1}$

Thus ℓ intersects the xy-plane at

$$\vec{r}\left(-\frac{1}{c-1}\right) = \left\langle -\frac{a}{c-1}, -\frac{b}{c-1}, 0\right\rangle,$$

Let $P' = \left(-\frac{a}{c-1}, -\frac{b}{c-1}, 0\right)$

(c) Let $f(x,y,z) = \left(\frac{-x}{z-1}, -\frac{y}{z-1}, 0\right)$

Then $f(P) = f(a,b,c) = \left(\frac{-a}{z-1}, \frac{-b}{z-1}, 0\right) = P'$

(d) Let $(x,y,0)$ be in the xy-plane.

Then

$$(f\circ g)(x,y,0) = f\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

$$= \left(\frac{-\frac{2x}{x^2+y^2+1}}{\frac{x^2+y^2-1}{x^2+y^2+1}-1}, \frac{\frac{-2y}{x^2+y^2+1}}{\frac{x^2+y^2-1}{x^2+y^2+1}-1}, 0\right)$$

$$= \left(\frac{\frac{-2x}{x^2+y^2+1}}{\frac{-2}{x^2+y^2+1}}, \frac{\frac{-2y}{x^2+y^2+1}}{\frac{-2}{x^2+y^2+1}}, 0\right) = (x,y,0)$$

Let $(x,y,z) \in S$. Note that $x^2+y^2+z^2=1$

$$(g\circ f)(x,y,z) = g\left(\frac{-x}{z-1}, \frac{-y}{z-1}, 0\right)$$

$$= \left(\frac{2\left(\frac{-x}{z-1}\right)}{\left(\frac{-x}{z-1}\right)^2+\left(\frac{-y}{z-1}\right)^2+1}, \frac{2\left(\frac{-y}{z-1}\right)}{\left(\frac{-x}{z-1}\right)^2+\left(\frac{-y}{z-1}\right)^2+1}, \frac{\left(\frac{-x}{z-1}\right)^2+\left(\frac{-y}{z-1}\right)^2-1}{\left(\frac{-x}{z-1}\right)^2+\left(\frac{-y}{z-1}\right)^2+1} \right)$$

$$= \left(\frac{\frac{-2x}{z-1}}{\frac{x^2+y^2+z^2-2z+1}{(z-1)^2}}, \frac{\frac{-2y}{z-1}}{\frac{x^2+y^2+z^2-2z+1}{(z-1)^2}}, \frac{\frac{x^2+y^2-z^2+2z-1}{(z-1)^2}}{\frac{x^2+y^2+z^2-2z+1}{(z-1)^2}} \right)$$

$$= \left(\frac{\frac{-2x}{z-1}}{\frac{-2z+2}{(z-1)^2}}, \frac{\frac{-2y}{z-1}}{\frac{-2z+2}{(z-1)^2}}, \frac{\frac{1-z^2-z^2+2z-1}{(z-1)^2}}{\frac{-2z+2}{(z-1)^2}} \right)$$

$$= \left(\frac{\frac{-2x}{z-1}}{\frac{-2(z-1)}{(z-1)^2}}, \frac{\frac{-2y}{z-1}}{\frac{-2(z-1)}{(z-1)^2}}, \frac{\frac{-2z(z-1)}{(z-1)^2}}{\frac{-2(z-1)}{(z-1)^2}} \right)$$

$$= (x,y,z).$$