Homework 3 Solutions

p.54:

2) Use the definition of limits to prove that $\lim_{z\to z_0}(az+b)=az_0+b$

Let $\varepsilon>0$. Choose $\delta=\frac{\varepsilon}{a}$.
If $|z-z_0|<\delta$, then
$$|f(z)-(az_0+b)|=|az+b-az_0-b|=a|z-z_0|<a\delta=a\frac{\varepsilon}{a}=\varepsilon$$

5.) Show $f(z)=\left(\frac{z}{\bar{z}}\right)^2$ has a value of 1 at all nonzero points on the real and imaginary axes, but it has a value of -1 at all points on the line $y=x$. Show that $\lim_{z\to 0} f(z)$ does not exist.

- On real axis, $z=x+i0$. Thus $\left(\frac{z}{\bar{z}}\right)^2=\left(\frac{x}{x}\right)^2=1$.
- On imaginary axis, $z=0+iy$. Thus $\left(\frac{z}{\bar{z}}\right)^2=\left(\frac{iy}{-iy}\right)^2=(-1)^2=1$
- On the line $y=x$, $z=x+ix$. Thus $\left(\frac{z}{\bar{z}}\right)^2=\left(\frac{x+ix}{x-ix}\right)^2=\left(\frac{x+ix}{x-ix}\cdot\frac{x+ix}{x+ix}\right)^2=\left(\frac{2x^2i}{2x^2}\right)^2=i^2=-1$

Since $\left(\frac{z}{\bar{z}}\right)^2\to 1$ as $z\to 0$ along the real/imaginary axes

and $\left(\frac{z}{\bar{z}}\right)^2\to -1$ as $z\to 0$ along the line $y=x$,

$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$ DNE.

11.) Use the theorem in Section 17 to show that if $T(z) = \frac{az+b}{cz+d}$, where $(ad-bc \neq 0)$, then

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c=0$

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -\frac{d}{c}} T(z) = \infty$ if $c \neq 0$.

(a) If $c=0$, then $T(z) = \frac{a}{d}z + \frac{b}{d}$. Since $ad \neq 0$, $a, d \neq 0$

Now $\lim_{z\to 0} \frac{1}{T(\frac{1}{z})} = \lim_{z\to 0} \frac{1}{\frac{a}{dz} + \frac{b}{d}} = \lim_{z\to 0} \frac{dz}{a+bz} = 0$ since $a \neq 0$.

By the theorem, $\lim_{z\to\infty} T(z) = \infty$

(b) • $\lim_{z\to 0} T(\frac{1}{z}) = \lim_{z\to 0} \frac{a(\frac{1}{z})+b}{c(\frac{1}{z})+d} = \lim_{z\to 0} \frac{a+bz}{c+dz} = \frac{a}{c}$

By the theorem, $\lim_{z\to\infty} T(z) = \frac{a}{c}$

• $\lim_{z\to -\frac{d}{c}} \frac{1}{T(z)} = \lim_{z\to -\frac{d}{c}} \frac{cz+d}{az+b} = \frac{0}{ad-bc} = 0$ since $ad-bc \neq 0$.

By the theorem, $\lim_{z\to -\frac{d}{c}} T(z) = \infty$.

p. 61

1.) Prove that if $w = z^2$, then $\frac{dw}{dz} = 2w$

$$\frac{dw}{dz} = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} 2z + \Delta z = 2z.$$

2.) Find the derivatives of (b) $f(z) = (2z^2+i)^5$, (d) $f(z) = \frac{(1+z^2)^4}{z^2}$

(b) $f'(z) = 5(2z^2+i)^4(4z) = 20z(2z^2+i)^4$

(d) $f'(z) = \dfrac{[4(1+z^2)^3 2z]z^2 - 2z(1+z^2)^4}{z^4}$

$= \dfrac{8z^3(1+z^2)^3 - 2z(1+z^2)^4}{z^4}$

$= \dfrac{2(3z^2-1)(1+z^2)^3}{z^3}$

8.) (a) Show that $f(z) = \mathrm{Re}(z)$ is not differentiable anywhere

Let $z = x+iy$, $z_0 = x_0 + iy_0$, $\Delta z = z - z_0 = (x-x_0) + i(y-y_0) = \Delta x + i\Delta y$

Then $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim\limits_{\Delta z \to 0} \dfrac{x_0 + \Delta x - x_0}{\Delta z} = \lim\limits_{\Delta z \to 0} \dfrac{\Delta x}{\Delta z}$

As $\Delta z \to 0$ along the line $\Delta x = 0$, $\dfrac{\Delta x}{\Delta z} = 0$

As $\Delta z \to 0$ along the line $\Delta y = 0$, $\dfrac{\Delta x}{\Delta z} = \dfrac{\Delta x}{\Delta x + i\Delta y} = \dfrac{\Delta x}{\Delta x} = 1$

Thus $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{\Delta x}{\Delta z}$ does not exist.

<u>p. 70</u>:

1.) Use the Cauchy-Riemann equations to show $f'(z)$ does not exist at any point if (a) $f(z) = \bar{z}$, (c) $f(z) = 2x + ixy^2$

(a) $f(z) = \bar{z} = x - iy$.

Let $u(x,y) = x$, $v(x,y) = -y$.

Since $u_x = 1 \neq -1 = v_y(x,y)$, the Cauchy-Riemann equations are not satisfied. Thus $f'(z)$ does not exist anywhere.

(c) $f(z) = 2x + ixy^2$

Let $u(x,y) = 2x$, $v(x,y) = xy^2$.

Then $u_x = 2$, $v_y = 2xy$, $u_y = 0$, $v_x = y^2$

Thus if f' exists, then ① $2 = 2xy$

② $0 = y^2$

Now ② $\Rightarrow y = 0$, which contradicts ①.

Thus f' never exists.

8.) (a) Let $F(x,y)$ be a function of 2 real variables and let $z = x+iy$.

Derive the formula $\dfrac{\partial F}{\partial \bar{z}} = \dfrac{\partial F}{\partial x}\dfrac{\partial x}{\partial \bar{z}} + \dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial F}{\partial x} + i\dfrac{\partial F}{\partial y}\right)$

Since $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$,

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i} = \frac{i}{2}$$

By the chain rule,

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial F}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial F}{\partial y}\left(\frac{i}{2}\right) = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

(b) Define $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$.

Show that if $f(z) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations, then $\dfrac{\partial f}{\partial \bar{z}} = 0$.

If f satisfies the C-R eqns, then $u_x = v_y$, $u_y = -v_x$

Thus

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\Big((u_x + iv_x) + i(u_y + iv_y)\Big)$$

$$= \frac{1}{2}\Big((u_x - v_y) + i(v_x + u_y)\Big) = \frac{1}{2}(0 + i(0)) = 0.$$