Homework 4 Solutions

p.70

1.) (d) Use Cauchy-Riemann equations to show $f(z) = e^x e^{-iy}$ is not differentiable anywhere.

$f(z) = e^x e^{-iy} = e^x \cos y - i e^x \sin y$

Let $u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$

Then $u_x = e^x \cos y$, $u_y = -e^x \sin y$

$v_x = -e^x \sin y$, $v_y = -e^x \cos y$

If $u_x = v_y$ and $u_y = -v_x$, then

$\cos y = -\cos y$ and $-\sin y = \sin y$

$\Rightarrow$ $\cos y = 0$ and $\sin y = 0$

$\Rightarrow$ $y = \frac{(2k+1)\pi}{2}$ and $y = n\pi$ for some $k, n \in \mathbb{Z}$.

Thus y is both an even and odd multiple of π, which is not possible.

Thus the Cauchy-Riemann equations are not satisfied at <u>any</u> points.

3.) Determine where $f'(z)$ exists and find its value when
(a) $f(z) = \frac{1}{z}$ (b) $f(z) = x^2 + iy^2$ (c) $f(z) = z \operatorname{Im} z$

(a) $\frac{1}{z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$

First note that $f(0)$ is undefined. Thus $f'(0)$ does not exist. Now suppose $z \neq 0$.

Let $u(x,y) = \frac{x}{x^2+y^2}$ and $v(x,y) = -\frac{y}{x^2+y^2}$.

Then $u_x = \frac{-x^2+y^2}{(x^2+y^2)^2}$, $u_y = -\frac{2xy}{(x^2+y^2)^2}$, $v_x = \frac{2xy}{(x^2+y^2)^2}$, $v_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$

and so $u_x = v_y$ and $u_y = -v_x$

Moreover, u_x, u_y, v_x, v_y all exist everywhere except $(0,0)$ and are continuous everywhere except $(0,0)$.

Thus, $f'(z)$ exists for all $z \neq 0$ and

$$f'(z) = u_x + i v_x = \frac{-x^2+y^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} = -\frac{(x-iy)^2}{((x+iy)(x-iy))^2} = \frac{-1}{(x+iy)^2} = \frac{-1}{z^2}$$

(b) $f(z) = x^2 + iy^2$

Let $u(x,y) = x^2$ and $v(x,y) = y^2$

Then $u_x = 2x$, $u_y = 0$, $v_x = 0$, $v_y = 2y$

If $x \neq y$, then $u_x \neq v_y$.

Thus f is not differentiable when $x \neq y$.

If $x = y$, then $u_x = v_y$ and $u_y = -v_x$.

Moreover the partial derivatives exist and are continuous everywhere. Thus $f'(z)$ exists whenever $x = y$ and $f'(z) = u_x + i v_x = 2x$

(c) $f(z) = z \operatorname{Im} z = (x+iy)y = xy + iy^2$

Let $u(x,y) = xy$, $v(x,y) = y^2$

Then $u_x = y$, $u_y = x$, $v_x = 0$, $v_y = 2y$

If $x \neq 0$ or $y \neq 0$, then the Cauchy-Riemann equations are not satisfied. Thus f is not differentiable when $z \neq 0$.

When $z = 0$, then $x = y = 0$ and so $u_x = v_y$, $u_y = -v_x$.

Moreover, u_x, u_y, v_x, v_y exist near $(0,0)$ and are continuous at $(0,0)$.

Thus $f'(0)$ exists and $f'(0) = u_x(0,0) + i v_x(0,0) = 0$.

4.) (a) Use polar coordinates to show $f(z) = \frac{1}{z^4}$ $(z \neq 0)$ is differentiable and find $f'(z)$.

$$f(z) = \frac{1}{z^4} = \frac{1}{r^4} e^{-i4\theta} = \frac{1}{r^4}\cos(4\theta) - \frac{1}{r^4}\sin(4\theta)$$

Let $u(r,\theta) = \frac{1}{r^4}\cos(4\theta)$, $v(r,\theta) = \frac{-1}{r^4}\sin(4\theta)$

then:

$$u_r = -\frac{4\cos(4\theta)}{r^5}, \quad u_\theta = \frac{-4\sin(4\theta)}{r^4}, \quad v_r = \frac{4\sin(4\theta)}{r^5}, \quad v_\theta = -\frac{4\cos(4\theta)}{r^4}$$

These exist and are continuous everywhere except $z=0$.

Moreover, $ru_r = v_\theta$ and $u_\theta = -rv_r$

Thus $f'(z)$ exists for all $z \neq 0$ and

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-\frac{4\cos(4\theta)}{r^5} + i\left(\frac{4\sin(4\theta)}{r^5}\right)\right)$$

$$= \frac{-4e^{-i\theta}}{r^5}(\cos 4\theta - i\sin 4\theta) = \frac{-4e^{-i\theta}}{r^5}e^{-i4\theta} = \frac{-4e^{-i5\theta}}{r^5}$$

$$= \frac{-4}{(re^{i\theta})^5} = \frac{-4}{z^5}$$

p. 76

1.) (a) Show that $f(z) = 3x + y + i(3y - x)$ is entire

Let $u(x,y) = 3x + y$, $v(x,y) = 3y - x$

Then $u_x = 3$, $u_y = 1$, $v_x = -1$, $v_y = 3$

Since $u_x = v_y$ and $u_y = -v_x$ at all points and u_x, v_x, u_y, v_y exist and are continuous everywhere, f is differentiable everywhere

Thus f is entire.

2.) (a) Show that $f(z) = xy + iy$ is nowhere analytic

Let $u(x,y) = xy$ and $v(x,y) = y$.
Then $u_x = y$, $u_y = x$, $v_x = 0$, $v_y = 1$
If $(x,y) \neq (0,1)$, then the Cauchy-Riemann equations are not satisfied. Thus $f'(z)$ does not exist when $z \neq i$. Thus f is not analytic when $z \neq i$

Since u_x, u_y, v_x, v_y exist near $(0,1)$ and are continuous at $(0,1)$, $f'(i)$ exists.
However, f is <u>not</u> analytic at i since f is not differentiable at all points near i

7.) Let f be analytic everywhere in a domain D.
Prove that if $f(z) \in \mathbb{R}$ for all $z \in D$, then $f(z)$ is constant throughout D.

Since $f(z) \in \mathbb{R}$, $f(z) = u(x,y) + iv(x,y)$, where $v(x,y) = 0$.
Since $f'(z)$ exists everywhere, the Cauchy-Riemann equations are satisfied: $u_x = v_y = 0$, $u_y = -v_x = 0$.
and $f'(z) = u_x + iv_x = 0$ everywhere on D.
Thus f is constant on D.

P. 95

4.) Show that $\log(i^2) \neq 2\log(i)$ on the branch $\log z = \ln r + i\theta$
$\left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$

$\log(i^2) = \log(-1) = \ln 1 + i(3\pi) = 3\pi i$
$2\log(i) = 2\left(\ln 1 + i\frac{5\pi}{2}\right) = 5\pi i$
Thus $\log(i^2) = 3\pi i \neq 5\pi i = 2\log(i)$ on this branch

p.107:

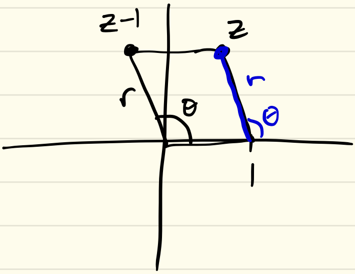
1.) Show that $\frac{d}{dz}(\sin z) = \cos z$ and $\frac{d}{dz}(\cos z) = -\sin z$

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{ie^{iz}-ie^{-iz}}{2} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z$$

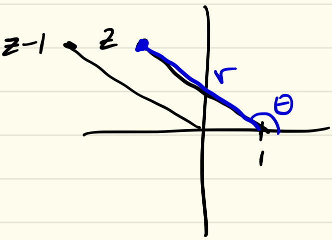
Addition Problems

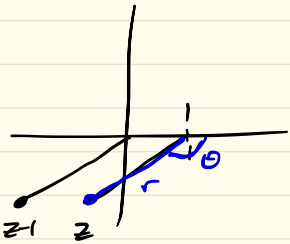
1.)(a)

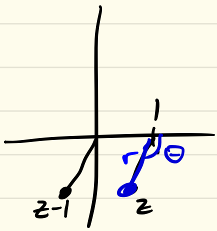


Let l_1 be the line from 0 to z and l_2 the line from 1 to z. By basic geometry, the lines are parallel, have the same length, r, and intersect the real axis at the same angle, θ

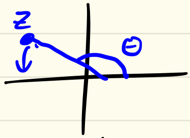
In the other quadrants, we have similar pictures



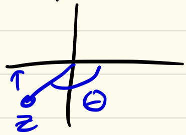




(b) As z approaches B from above the real axis, $\theta \to \pi$



As z approaches B from below the real axis, $\theta \to -\pi$



(c) Since $F(z) = \ln|z-1| + i\theta \to \ln|z-1| + i\pi$ as $z \to B$ from above and $F(z) \to \ln|z-1| - i\pi$ as $z \to B$ from below, F is not continuous on B.

(d) $F(z) = \ln r + i\theta$, where $-\pi < \theta < \pi$, $r > 0$.

let $u(r,\theta) = \ln r$, $v(r,\theta) = \theta$.

Then $u_r = \frac{1}{r}$, $u_\theta = 0$, $v_r = 0$, $v_\theta = 1$

Thus

$$r u_r = 1 = v_\theta \quad \text{and} \quad u_\theta = 0 = -r v_r$$

Moreover, since $u_r, u_\theta, v_r, v_\theta$ exist and are continuous everywhere on U, F is differentiable everywhere on U.

Thus F is analytic on U.

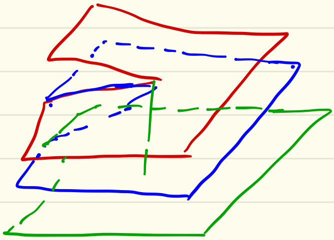
(e) Since F is analytic on U, F is a branch of F.

Since B is a curve in $\mathbb{C}$ and F is a branch on $\mathbb{C}-B$, B is a branch cut, by definition.

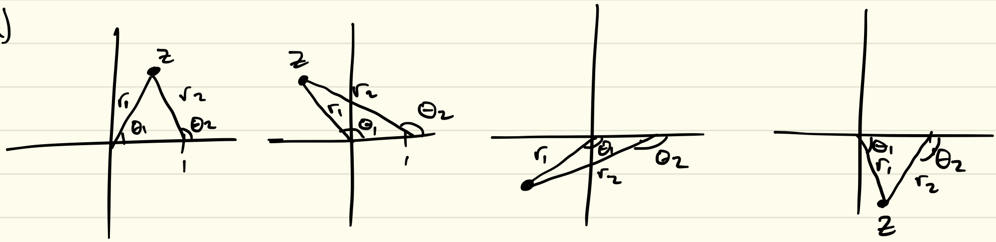
Since f is undefined at $z=1$, $z=1$ is in every branch cut of f.

Thus $z=1$ is a branch point.

(f) An attempt:



2.) (a)



(b) As $z \to B$ from above, $\theta_1 \to 0$ and $\theta_2 \to \pi \Rightarrow \theta_1 - \theta_2 \to \pi$
As $z \to B$ from below, $\theta_1 \to 0$ and $\theta_2 \to -\pi \Rightarrow \theta_1 - \theta_2 \to -\pi$
As $z \to B'$ from above or below, $\theta_1 \to 0$ and $\theta_2 \to 0 \Rightarrow \theta_1 - \theta_2 \to 0$
As $z \to B''$ from above or below, $\theta \to \pi$ and $\theta_2 \to \pi \Rightarrow \theta_1 - \theta_2 \to 0$

(c) Since $G(z) \to \ln|z-1| + i\pi$ as $z \to B$ from above
and $G(z) \to \ln|z-1| - i\pi$ as $z \to B$ from below,
G is not continuous on B.

(e) No way.