Homework 5 Solutions:

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5.) Suppose f is analytic at $z_0 = z(t_0)$, which lies on a smooth arc $z = z(t)$, $a \le t \le b$.
Let $w(t) = f(z(t))$. Show that $w'(t_0) = f'(z(t_0))\, z'(t_0)$

Let $f(z) = u(x,y) + iv(x,y)$, $z(t) = x(t) + iy(t)$
Since f is analytic at $z_0 = z(t_0)$,
$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$$

Now
$$w(t) = u(x(t), y(t)) + iv(x(t), y(t))$$
$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y')$$
$$= x'(u_x + iv_x) + y'(u_y + iv_y)$$
$$= x'(u_x + iv_x) + iy'(v_y - iu_y)$$
Thus
$$w'(t_0) = x'(t_0) f'(z_0) + iy'(t_0) f'(z_0)$$
$$= f'(z_0)(x'(t_0) + iy'(t_0)) = f'(z(t_0))\, z'(t_0).$$

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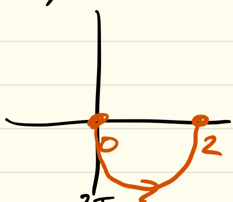
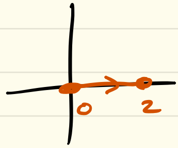
2.) Let $f(z) = z - 1$. Compute $\int_C f(z)\,dz$, where
a) C is given by $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$
b) C is given by $z = x$, $(0 \le x \le 2)$

a) C can be parametrized by
$z(t) = 1 + e^{it}$, $\pi \le t \le 2\pi$
$z'(t) = ie^{it}$



$$\int_C f(z)\,dz = \int_\pi^{2\pi} (1 + e^{it} - 1)\, ie^{it}\,dt = \int_\pi^{2\pi} ie^{2it}\,dt$$
$$= \int_\pi^{2\pi} -\sin 2t\,dt + i\int_\pi^{2\pi} \cos 2t\,dt = 0$$

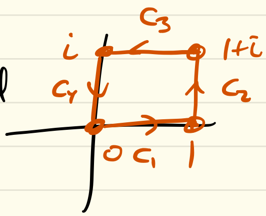
b) Parametrize C by $z(t) = t$, $0 \le t \le 2$
Then $z'(t) = 1$.



$$\int_C f(z)\,dz = \int_0^2 (t-1)\,dt = 0$$

3.) Let $f(z) = \pi e^{\pi \bar{z}}$ and let C be the boundary of the square with vertices at $0, 1, 1+i$, and i oriented counterclockwise. Calculate $\int_C f(z)\,dz$

Let C_1, C_2, C_3, C_4 be as in the picture. These can be parametrized by



$z_1(t) = t$, $0 \le t \le 1$,
$z_2(t) = 1 + it$, $0 \le t \le 1$
$z_3(t) = (1-t) + i$ $\quad 0 \le t \le 1$
$z_4(t) = -it$, $0 \le t \le 1$

Thus $z_1'(t) = 1$, $z_2'(t) = i$, $z_3'(t) = -1$, $z_4'(t) = -i$

Now,
$$\int_{C_1} f(z)\,dz = \int_0^1 \pi e^{\pi t}\,dt = e^{\pi} - 1$$

$$\int_{C_2} f(z)\,dz = \int_0^1 \pi e^{\pi(1-it)} \cdot i\,dt = \pi i e^{\pi} \int_0^1 e^{-i\pi t}\,dt = 2e^{\pi}$$

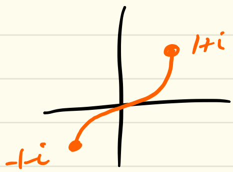
$$\int_{C_3} f(z)\,dz = \int_0^1 \pi e^{\pi(1-t-i)}(-1)\,dt = -\pi e^{\pi} e^{-\pi i} \int_0^1 e^{-\pi t}\,dt = e^{\pi} - 1$$

$$\int_{C_4} f(z)\,dz = \int_0^1 \pi e^{\pi i t}\,dt = e^{i\pi} - 1 = -2$$

$$\Rightarrow \int_C f(z)\,dz = e^{\pi} - 1 + 2e^{\pi} + e^{\pi} - 1 - 2 = 4e^{\pi} - 4.$$

4.) Let $f(z) = \begin{cases} 1 & \text{if } y < 0 \\ 4y & \text{if } y > 0 \end{cases}$

Let C be the arc of $y = x^3$ from $z = -1-i$ to $z = 1+i$.
Calculate $\int_C f(z)\,dz$



We can parametrize C by
$z(t) = t + it^3, \quad -1 \leq t \leq 1$
Thus $z'(t) = 1 + 3t^2 i$.

Since $f(z(t))$ has a discontinuity when $t = 0$, we need to break this into 2 parts.
Let C_1 be the portion of C where $y<0$, and let C_2 be the other
When $y < 0$, $f(z) = 1$

$$\int_{C_1} f(z)\,dz = \int_0^1 1(1+3t^2 i)\,dt = \int_0^1 dt + i\int_0^1 3t^2\,dt = 1 + i.$$

When $y > 0$, $f(z) = 1 \Rightarrow f(z(t)) = 4t^3$

$$\int_{C_2} f(z)\,dz = \int_0^1 4t^3(1+3t^2 i)\,dt = \int_0^1 4t^3\,dt + i\int_0^1 12t^6\,dt = 1+2i$$

Thus $\int_C f(z)\,dz = 1+i + 1+2i = 2+3i$.

5.) Let $f(z) = 1$ and C be any contour from z_1 to z_2.
Calculate $\int_C f(z)\,dz$.

Let $z(t)$, $a \leq t \leq b$, be a parametrization of C, where $z(a) = z_1$ and $z(b) = z_2$. Then

$$\int_C f(z)\,dz = \int_a^b 1 \cdot z'(t)\,dt = \int_a^b z'(t)\,dt = z(b) - z(a) = z_2 - z_1,$$

by the fundamental theorem of Calculus.

13.) Let C_0 be the circle parametrized by $z = z_0 + Re^{i\theta}$, $-\pi \le \theta \le \pi$
Show that

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \begin{cases} 0 & n = \pm 1, \pm 2, \ldots \\ 2\pi i & n = 0 \end{cases}$$

Since $z(t) = z_0 + Re^{it}$, $z'(t) = Rie^{it}$
then

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \int_{-\pi}^{\pi} (Re^{it})^{n-1} \cdot Rie^{it}\,dt$$

$$= \int_{-\pi}^{\pi} R^n i e^{int}\,dt = R^n \int_{-\pi}^{\pi} ie^{int}\,dt$$

If $n=0$, then we have $\int_{-\pi}^{\pi} i\,dt = 2\pi i$
If $n \neq 0$, we have

$$R^n \int_{-\pi}^{\pi} ie^{int}\,dt = R^n \int_{-\pi}^{\pi} -\sin nt\,dt + R^n i \int_{-\pi}^{\pi} \cos nt\,dt$$

$$= R^n(0) + R^n i(0) = 0$$

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1.) Use an antiderivative to show if C is any contour from z_1 to z_2, then $\int_C z^n = \frac{1}{n+1}(z_2^{n+1} - z_2^n)$ $(n \in \mathbb{Z}^+)$

Since $F(z) = \frac{1}{n+1} z^{n+1}$ is an antiderivative of $f(z) = z^n$,

$$\int_C z^n\,dz = F(z_2) - F(z_1) = \frac{1}{n+1} z_2^{n+1} - \frac{1}{n+1} z_2^{n+1}$$

2.) Evaluate the following by finding antiderivatives.

a) $\int_0^{1+i} z^2\,dz$, b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)dz$ c) $\int_1^3 (z-2)^3\,dz$

a) $\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{(1+i)^3}{3.} = \frac{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^3}{3} = \frac{2\sqrt{2}e^{i\frac{3\pi}{4}}}{}$

$= \frac{2\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}{3} = \frac{2}{3}(-1+i)$

b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)dz = 2\sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2i} = 2\sin\left(\frac{\pi+2i}{2}\right)$

$= 2\left(\frac{e^{i\frac{\pi+2i}{2}} - e^{-i\frac{\pi+2i}{2}}}{2i}\right)$

$= \frac{e^{\frac{i\pi}{2}}e^{-1} - e^{-\frac{i\pi}{2}}e^{1}}{i} = \frac{ie^{-1}+ie}{i} = e+e^{-1}$

c) $\int_1^3 (z-2)^3\,dz = \frac{1}{4}(z-2)^4\Big|_1^3 = 0$

3.) Show $\int_{C_0}(z-z_0)^{n-1}\,dz = 0$ $(n=\pm1,\pm2,\ldots)$ for all closed contours that do not pass through z_0.

When $n\geq 1$, $f(z)=(z-z_0)^{n-1}$ is analytic on $\mathbb{C}$

When $n\leq -1$, $f(z)$ is analytic on $\mathbb{C}-\{z_0\}$.

Since C_0 is a closed contour in $\mathbb{C}-\{z_0\}$, and f has an antiderivative in $\mathbb{C}-\{z_0\}$, namely $F(z)=\frac{1}{n}(z-z_0)^n$, $\int_{C_0}(z-z_0)^{n-1}\,dz = 0$, by the theorem.