# Homework 6 Solutions

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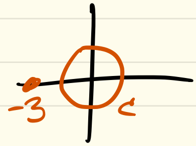
1.) Apply Cauchy-Goursat to show $\int_C f(z)\,dz = 0$ when $C$ is the unit circle and

(a) $f(z) = \dfrac{z^3}{z+3}$ (b) $f(z) = ze^{-z}$ (c) $f(z) = \dfrac{1}{z^2+2z+2}$

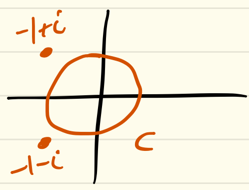
(a) $f(z) = \dfrac{z^3}{z+3}$ is analytic everywhere except $z=-3$.




Since $C$ does not enclose or pass through $z=-3$, $f$ is analytic at all points on $C$ and enclosed by $C$. Thus by Cauchy-Goursat, $\int_C \dfrac{z^3}{z+3}\,dz = 0$

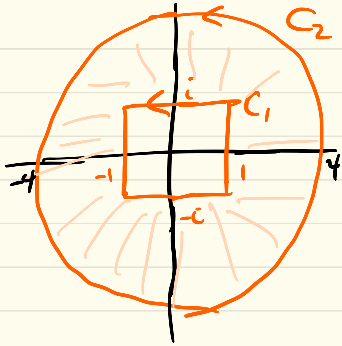
(b) $f(z) = ze^{-z}$ is analytic on $\mathbb{C}$. Thus, $f$ is analytic at all points on $C$ and enclosed by $C$. Thus by Cauchy-Goursat, $\int_C ze^{-z}\,dz = 0$

(c) $f(z) = \dfrac{1}{z^2+2z+2}$ is analytic everywhere except when $z^2+2z+2=0$
$\Rightarrow z = -1 \pm i$




Since $C$ does not enclose or pass through $-1\pm i$, $f$ is analytic at all points on $C$ and enclosed by $C$. Thus $\int_C \dfrac{1}{z^2+2z+2}\,dz = 0$.

2) Let $C_1$ and $C_2$ be the square and circle depicted.




Explain why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$ (b) $f(z) = \dfrac{z+2}{\sin(z/2)}$

(a) $f(z) = \dfrac{1}{3z^2+1}$ is analytic at all points except when $3z^2+1=0 \Rightarrow z = \pm\frac{\sqrt{3}}{3}i$

Since $\left|\pm\frac{\sqrt{3}}{3}i\right| < 1$, $\pm\frac{\sqrt{3}}{3}i$ are not in the region between $C_1$ and $C_2$ (including $C_1$ & $C_2$), $f$ is analytic on $C_1, C_2$ and the region.

Thus $\displaystyle\int_{C_1} \frac{1}{3z^2+1}\,dz = \int_{C_2} \frac{1}{3z^2+1}\,dz$

(b) $f(z) = \dfrac{z+2}{\sin(\frac{z}{2})}$ is analytic everywhere except at $z = 2n\pi$, $n \in \mathbb{Z}$

Since these points do not lie in the region between $C_1$ and $C_2$ or on $C_1$ or $C_2$, $f$ is analytic at all points on $C_1$ and $C_2$ and between $C_1$ and $C_2$.

Thus $\displaystyle\int_{C_1} \frac{z+2}{\sin(\frac{z}{2})}\,dz = \int_{C_2} \frac{z+2}{\sin(\frac{z}{2})}\,dz$

7.) Show that if $C$ is a positively oriented simple closed contour, then the area of the region enclosed by $C$ is $\frac{1}{2i}\int_C \bar{z}\,dz$

Let $R$ be the region enclosed by $C$.
First, we can write $f(z) = \bar{z} = x - iy$. So $u(x,y) = x$, $v(x,y) = -y$.
By formula (4) on page 149,

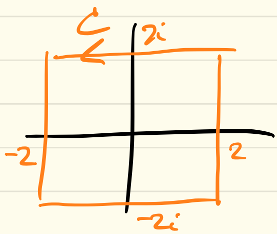
$$\int_C \bar{z}\,dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA$$

$$= \iint_R 0\,dA + i\iint_R 1-(-1)\,dA = i\iint_R 2\,dA$$

Thus $\frac{1}{2i}\int_C \bar{z}\,dz = \frac{1}{2i}\,i\iint_R 2\,dA = \iint_R 1\,dA = \text{Area of } R.$

p.170

1.) Let $C$ be the positively oriented boundary of the square depicted.
Evaluate:




(a) $\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\,dz$ (b) $\int_C \frac{\cos z}{z(z^2+8)}\,dz$ (c) $\int_C \frac{z\,dz}{2z+1}$

(d) $\int_C \frac{\cosh z}{z^4}\,dz$ (e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz$ $(-2 < x_0 < 2)$

(a) Let $f(z)=e^{-z}$. Then $f$ is analytic everywhere.
Since $\frac{\pi}{2}i$ is enclosed by $C$, by the Cauchy Integral formula,

$$\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}\,dz = 2\pi i f\left(\tfrac{\pi}{2}i\right) = 2\pi i e^{-\frac{\pi i}{2}} = 2\pi$$

(b) Let $f(z)=\frac{\cos z}{z^2+8}$. Then $f$ is analytic everywhere except $z=\pm\sqrt{8}i$. Since these points are not on or interior to $C$, and since $z=0$ is interior to $C$, by Cauchy Integral formula,

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i f(0) = 2\pi i\left(\tfrac{1}{8}\right) = \frac{\pi i}{4}$$

(c) Let $f(z)=\frac{z}{2}$. Then $f$ is analytic inside and on $C$
Since $z=-\frac{1}{2}$ is enclosed by $C$, by the Cauchy Integral formula,

$$\int_C \frac{z}{2(z+\frac{1}{2})}\,dz = 2\pi i f(-\tfrac{1}{2}) = -\frac{\pi i}{2}$$

(d) Let $f(z)=\cosh z$. Then $f$ is analytic inside and on $C$
Since $z=0$ is enclosed by $C$, by the Cauchy Integral derivative formula

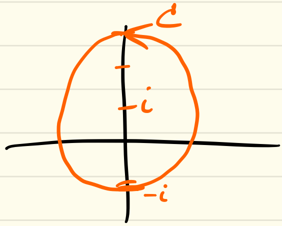
$$\int_C \frac{\cosh z}{z^4}\,dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{\pi i}{3}\sinh(0) = 0$$

(e) Let $f(z)=\tan(\frac{z}{2})$. Then $f$ is analytic inside and on $C$
Since $z=x_0$ is enclosed by $C$, by the Cauchy Integral derivative formula

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = \frac{2\pi i}{1!} f'(x_0) = 2\pi i\left(\tfrac{1}{2}\sec^2\left(\tfrac{x_0}{2}\right)\right) = \pi i \sec^2\left(\tfrac{x_0}{2}\right)$$

2.) Let $C$ be the circle $|z-i|=2$ oriented positively.
Evaluate

(a) $\int_C \frac{1}{z^2+4}\,dz$ (b) $\int_C \frac{1}{(z^2+4)^2}\,dz$




(a) Let $f(z)=\frac{1}{z+2i}$. Then $f$ is analytic everywhere except $z=-2i$.
Since $-2i$ is not enclosed by $C$ or on $C$, $f$ is analytic on $C$ and in $C$.
Thus

$$\int_C \frac{1}{z^4+4}\,dz = \int_C \frac{\frac{1}{z+2i}}{z-2i}\,dz = 2\pi i f(2i) = \frac{\pi}{2}$$

(b) Let $f(z)=\frac{1}{(z+2i)^2}$. Then $f$ is analytic on $C$ and inside $C$ since $-2i$ is not inside or on $C$.
Thus

$$\int_C \frac{1}{(z^2+4)^2}\,dz = \int_C \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2}\,dz = 2\pi i f'(2i) = 2\pi i\left(\frac{-2}{(2i+2i)^3}\right)$$
$$= 2\pi i\left(\frac{-2}{-64i}\right) = \frac{\pi}{16}$$

3.) Let $C$ be the circle $|z|=3$ oriented positively. Let $g(z)=\int_C \frac{2s^2-s-2}{s-z}\,dz$ $(|z|\neq 3)$

Show that $g(2)=8\pi i$. What is $g(z)$ when $|z|>3$?

Let $f(s)=2s^2-s-2$. Since $f$ is analytic on $C$ and $2$ is enclosed by $C$, by the Cauchy Integral formula,

$$g(2)=\int_C \frac{2s^2-s-2}{s-2}\,ds = 2\pi i f(2) = 2\pi i(2(2)^2-2-2) = 8\pi i$$

If $|z|>3$, then the function $h(s)=\frac{2s^2-s-2}{s-z}$ is analytic on and inside of $C$.
Thus by the Cauchy-Goursat Theorem, $g(z)=\int_C \frac{2s^2-s-2}{s-z}ds=0$

4.) Let $C$ be any positively oriented simple closed contour and let $g(z)=\int_C \frac{s^3+2s}{(s-z)^3}ds$.

Show that $g(z)=\begin{cases} 6\pi i z & \text{if } z \text{ is inside } C \\ 0 & \text{if } z \text{ is outside of } C \end{cases}$

If $z$ is inside $C$, let $f(s)=s^3+2s$. Then $f$ is analytic on $C$ and so by the Cauchy Integral Derivative formula,

$$g(z)=\int_C \frac{s^3+2s}{(s-z)^3}ds=\frac{2\pi i}{2!}f''(z)=\pi i(6z)=6\pi i z$$

If $z$ is outside $C$, then $\frac{s^3+2s}{(s-z)^3}$ is analytic on $C$ and inside $C$. Thus by the Cauchy-Goursat theorem, $g(z)=\int_C \frac{s^3+2s}{(s-z)^3}ds=0$.

5.) Show that if $f$ is analytic on and within a simple closed contour $C$ and $z_0$ is not on $C$, then $\int_C \frac{f'(z)}{z-z_0}dz=\int_C \frac{f(z)}{(z-z_0)^2}dz$

Since $f$ is analytic on and inside $C$, so is $f'$.
By the Cauchy Integral formula & derivative formula,

$$\int_C \frac{f'(z)}{z-z_0}dz=2\pi i f'(z_0) \quad \text{and} \quad f'(z_0)=\frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

Thus $\int_C \frac{f'(z)}{z-z_0}dz=\int_C \frac{f(z)}{(z-z_0)^2}dz$

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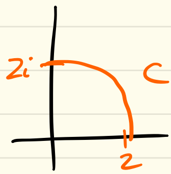
1.) Show $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \leq \frac{6\pi}{7}$ where C is the arc of $|z|=2$ from 2 to $2i$ in the $1^{st}$ quadrant.

Since $z$ is on C, $|z|=2$.

Thus $|z+4| \leq |z|+|4| = 2+4=6$

and $|z^3-1| \geq \big||z|^3-|1|\big| = 2^3-1=7$

$\Longrightarrow \left|\frac{z+4}{z^3-1}\right| \leq \frac{6}{7}$

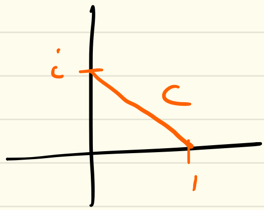



Moreover, $\text{length}(C) = \frac{2\pi(2)}{4} = \pi$

Thus $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \leq 6\,\text{length}(C) = \frac{6\pi}{7}$

2.) Let C be the line segment from $i$ to 1

Show that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$




Since $z$ is on C, $|z|$ is minimized at the midpoint of C, namely $\frac{1}{2}+\frac{1}{2}i$. Thus on C, $|z| \geq \frac{1}{\sqrt{2}}$

Thus $\left|\frac{1}{z^4}\right| = \frac{1}{|z|^4} \leq 4.$

Now the length of C is $|i-1| = \sqrt{1^2+1^2} = \sqrt{2}$

Thus $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$

4.) Let $C_R$ be the upper half of $|z|=R$ $(R>2)$

Show
$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \le \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

Then show $\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} \to 0$ as $R \to \infty$

Since $z$ is on $C$, $|z|=R$

Thus $|2z^2-1| \le 2|z|^2+|-1| = 2R^2+1$

$$|z^4+5z^2+4| = |(z^2+1)(z^2+4)| = |z^2+1||z^2+4|$$
$$\ge (||z|^2-1|)(||z^2|-4|) = (R^2-1)(R^2-4)$$

$$\Rightarrow \left|\frac{2z^2-1}{z^4+5z^2+4}\right| \le \frac{2R^2+1}{(R^2-1)(R^2-4)}$$

Now, Length of $C$ is $\pi R$

Thus $$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \le \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

Now $$\lim_{R\to\infty} \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)} = 0$$

$$\Rightarrow \lim_{R\to\infty}\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| = 0$$

$$\Rightarrow \lim_{R\to\infty}\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz = 0$$