Homework 7 Solutions

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10.) Let f be entire such that $|f(z)| \le A|z|$ for all z, where A is a positive real number.
Show that $f(z) = a_1 z$, where $a_1 \in \mathbb{C}$.

Let $z_0 \in \mathbb{C}$ and let C_R be a circle of radius R centered at z_0.
Since $R = |z - z_0| \ge |z| - |z_0|$, we have $|z| \le R + |z_0|$
Thus $|f(z)| \le A|z| \le A(|z_0| + R)$ for all z on C_R.
Now by Cauchy's Inequality, $|f''(z_0)| \le \frac{2A(|z_0|+R)}{R^2}$

As R gets very large, $\frac{2A|z_0|+R}{R^2}$ gets very small.

Thus $|f''(z_0)| \le \lim_{R\to\infty} \frac{2A(|z_0|+R)}{R^2} = 0$

$\Rightarrow$ $|f''(z_0)| = 0$ for all $z_0 \in \mathbb{C}$
$\Rightarrow$ $f''(z_0) = 0$ for all $z_0 \in \mathbb{C}$.

Since $f''(z) = 0$ $\forall z \in \mathbb{C}$, $f'(z)$ is a constant function, call it $f'(z) = a_1$.
Any antiderivative of $f'(z) = a_1$ is of the form $F(z) = a_1 z + b$, where $b \in \mathbb{C}$.
Now, $|F(z)| = |a_1 z + b|$.
If $|a_1 z + b| \le A|z|$ for all z, then when $z = 0$, $|b| \le 0$
Thus $b = 0 \Rightarrow f$ is of the form $f(z) = a_1 z$.

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1.) Let f be entire and let $u(x,y) = \text{Re}(f(z)) \leq u_0$ for all (x,y) in $\mathbb{R}^2$. Show that u is constant on $\mathbb{R}^2$.

Let $g(z) = e^{f(z)} = e^{u(x,y)}e^{iv(x,y)}$, where $f(z) = u(x,y) + iv(x,y)$.
Then $|g(z)| = e^{u(x,y)} \leq e^{u_0}$ for all (x,y) on $\mathbb{R}^2$.
By Liouville's Theorem, since g is entire on $\mathbb{C}$ and $g(z)$ is bounded on $\mathbb{C}$, g is constant on $\mathbb{C}$. Let $g(z) = e^{u(x,y)}e^{iv(x,y)} = c$, where $c \in \mathbb{C}$ is constant.
Then $e^{u(x,y)} = |c| \Rightarrow u(x,y) = \ln|c|$, which is constant on $\mathbb{R}^2$.

2.) Let f be continuous on a closed, bounded region R and let it be analytic and nonconstant on the interior of R. If $f(z) \neq 0$ on R, prove that $|f(z)|$ has a minimum value m in R occurring on the boundary of R and not on the interior.

Let $g(z) = \frac{1}{f(z)}$. Since g is nonconstant and analytic on R, by the Maximum Modulus Principle, $|g(z)|$ has no maximum in the interior of R.
Thus $|f(z)| = \frac{1}{|g(z)|}$ has no minimum in the interior of R.

8.) Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ $(a_n \neq 0)$ and let $P(z_0) = 0$. We will show $P(z) = (z - z_0)Q(z)$, where Q is a polynomial of degree $n-1$

a) Verify that $z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z z_0^{k-2} + z_0^{k-1})$ for all $k \geq 2$

$$(z-z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + zz_0^{k-2} + z_0^{k-1})$$
$$= z^k + \cancel{z^{k-1}z_0} + \cancel{z^{k-2}z_0^2} + \cdots + \cancel{zz_0^{k-1}} + \cancel{zz_0^{k-1}}$$
$$- \cancel{z_0 z^{k-1}} - \cancel{z^{k-2}z_0^2} - \cdots - \cancel{zz_0^{k-1}} - z_0^k$$
$$= z^k - z_0^k$$

b) Show that $P(z) - P(z_0) = (z - z_0)Q(z)$ where $Q(z)$ has degree $n-1$ and deduce the result.

$$P(z) - P(z_0) = a_0 + a_1 z + \cdots + a_n z^n - (a_0 + a_1 z_0 + \cdots + a_n z_0^n)$$
$$= a_1(z - z_0) + a_2(z^2 - z_0^2) + \cdots + a_n(z^n - z_0^n)$$
$$= a_1(z-z_0) + a_2(z-z_0)(z+z_0) + a_3(z-z_0)(z^2 + zz_0 + z_0^2)$$
$$+ \cdots + a_n(z-z_0)(z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1})$$
$$= (z-z_0)\Big[a_1 + a_2(z+z_0) + a_3(z^2 + 2z_0 + z_0^2) + \cdots + a_n(z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1})\Big]$$

Let $Q(z) = a_1 + a_2(z+z_0) + a$ [illegible] $z_0 + z_0^2) + \cdots +$
$a_n($ [illegible] $+ z$ [illegible] $^2 z_0 + \cdots + zz_0^{n-2} + z_0^{n-1})$

Then since $a_n \neq 0$, Q is a polynomial of degree $n-1$ and $P(z) - P(z_0) = (z - z_0)Q(z)$

$\Rightarrow P(z) = (z - z_0)Q(z)$, since $P(z_0) = 0$.

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1.)(a) Let $u(x,y) = 2x - x^3 + 3xy^2$. Show u is harmonic in some domain D and find a harmonic conjugate of u in D.

$$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$$
$$u_{xx} = -6x \qquad u_{yy} = 6x$$

Thus $u_{xx} + u_{yy} = 0$ on $\mathbb{C}$ and so u is harmonic on $\mathbb{C}$.

Let $v(x,y)$ be a harmonic conjugate of u on $\mathbb{C}$. Then $f(z) = u(x,y) + iv(x,y)$ is analytic and so $v_x = -u_y = -6xy$ and $v_y = u_x = 2 - 3x^2 + 3y^2$

Thus $v(x,y) = \int -6xy \, dx = -3x^2y + C_1(y)$

and
$$v(x,y) = \int 2 - 3x^2 + 3y^2 \, dy = 2y - 3x^2y + y^3 + C_2(x)$$

$\Rightarrow$ $C_1(y) = 2y + y^3 + C$ for some constant C

$\Rightarrow$ $v(x,y) = 2y + y^3 - 3x^2y + C$.

Thus, in particular, $v(x,y) = 2y + y^3 - 3x^2y$ is a harmonic conjugate of u.

3.) Suppose v is a harmonic conjugate of u on a domain D. If u is also a harmonic conjugate of v on D, show that u and v are constant on D.

By definition, the functions $f_1(z) = u(x,y) + iv(x,y)$ and $f_2(z) = v(x,y) + iu(x,y)$ are analytic on D. Thus the Cauchy-Riemann equations satisfied for both functions: $u_x = v_y$ and $v_x = u_y$

$v_x = -u_y$ $\quad$ $v_y = -u_x$

Thus $u_x = v_y = -u_x \Rightarrow 2u_x = 0 \Rightarrow u_x = 0$

$v_x = -u_y = -v_x \Rightarrow 2v_x = 0 \Rightarrow v_x = 0$

and so $f'(z) = u_x + iv_x = 0$ on D.

Thus f is constant on D.

Additional Problems

1.) a) $n(C;0) = 2$ $\quad$ $n(C;1) = 0$ $\quad$ $n(C;i) = -1$

b) Let $f(z) = \cos(i\pi z)$. Then f is analytic on $\mathbb{C}$. By the general Cauchy Integral Formula,

$$\int_C \frac{\cos(i\pi z)}{z - z_0}\,dz = n(C;z_0)\,2\pi i\, f(z_0)$$

If $z_0 = 0$, we have $\int_C \frac{\cos(i\pi z)}{z-z_0}\,dz = 2(2\pi i \cos(0)) = 4\pi i$

If $z_0 = 1$, we have $\int_C \frac{\cos(i\pi z)}{z-z_0}\,dz = 0(2\pi i)(\cos(i\pi)) = 0$

If $z_0 = i$, we have $\int_C \frac{\cos(i\pi z)}{z-z_0}\,dz = -1(2\pi i)(\cos(-\pi)) = 2\pi i$

2.) By the Cauchy Inequality, since f is analytic on and inside C, which is centered at $1+i$

$$|f'''(1+i)| \le \frac{3!\,(7)}{3^3} = \frac{14}{9}$$

3) Let z_0 be a point enclosed by C_R
Then by the Cauchy Integral Formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_R} \frac{c}{z-z_0}\,dz$$

$$= \frac{c}{2\pi i}\int_{C_R} \frac{1}{z-z_0}\,dz$$

By another application of the Cauchy Integral formula, $\displaystyle\int_{C_R} \frac{1}{z-z_0}\,dz = 2\pi i(1) = 2\pi i$

Thus $f(z_0) = \frac{c}{2\pi i}(2\pi i) = c$

$\Longrightarrow$ $f(z) = c$ for all z in $|z| \le R$.