Homework 8 Solutions

1.) Let $g(z) = \frac{1}{f(z)}$. Then since $|f(z)| \geq 4$ for all $z \in \mathbb{C}$,

$f(z) \neq 0$ for all $z \in \mathbb{C}$. Thus since f is entire, so is g.

Moreover, $|g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{4}$ for all $z \in \mathbb{C}$

$\Rightarrow$ g is bounded on $\mathbb{C}$.

Thus by Liouville's Thm, g is constant on $\mathbb{C}$

$\Rightarrow$ f is also constant on $\mathbb{C}$.

2.)

a) Since f is entire and not constant, $|f(z)|$ has no maximum value on $\mathbb{C}$.

b) $f(z) = z + e^z = x + iy + e^x e^{iy} = (x + e^x \cos y) + i(y + e^x \sin y)$

c) (1) Let (x_0, y_0) be a critical point of u. Then

- If $u_{xx}(x_0,y_0)\,u_{yy}(x_0,y_0) - u_{xy}(x_0,y_0) > 0$ and $u_{xx}(x_0,y_0) > 0$, then u has a local min at (x_0,y_0)
- If $u_{xx}(x_0,y_0)\,u_{yy}(x_0,y_0) - u_{xy}(x_0,y_0) > 0$ and $u_{xx}(x_0,y_0) < 0$ then u has a local max at (x_0,y_0)
- If $u_{xx}(x_0,y_0)\,u_{yy}(x_0,y_0) - u_{xy}(x_0,y_0) < 0$, then u has a saddle point at (x_0,y_0)

(2) $u_x = 1 + e^x \cos y = 0$

$u_y = -e^x \sin y = 0 \Rightarrow \sin y = 0 \Rightarrow y = \pi n, \; n \in \mathbb{Z}$

Plugging $y = n\pi$ into u_x gives $1 \pm e^x = 0 \Rightarrow e^x = \pm 1$

Since $e^x > 0$, $e^x = 1$ (which occurs when $y = (2n+1)\pi$)

$\Rightarrow x - 0$

Thus the critical points of f are $(0, (2n+1)\pi)$, $n \in \mathbb{Z}$.

Now,

$$\begin{aligned} u_{xx} &= e^x \cos y \\ u_{xy} &= -e^x \sin y \\ u_{yx} &= -e^x \sin y \\ u_{yy} &= -e^x \cos y \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_{xx}(0,(2n+1)\pi) &= -1 \\ u_{xy}(0,(2n+1)\pi) &= 0 \\ u_{yx}(0,(2n+1)\pi) &= 0 \\ u_{yy}(0,(2n+1)\pi) &= 1 \end{aligned}$$

$\Rightarrow$ at $(0,(2n+1)\pi)$, $u_{xx}u_{yy} - u_{xy}^2 = -1 < 0$

$\Rightarrow$ u has a saddle point at all critical points of u.

3) (a) If (x_0, y_0) is a critical point of u, then $u_x(x_0,y_0) = u_y(x_0,y_0) = 0$

Since $f'(z_0)$ exists, by the Cauchy-Riemann equations, $v_x(x_0,y_0) = -u_y(x_0,y_0) = 0$

and $v_y(x_0,y_0) = u_x(x_0,y_0) = 0$

Thus (x_0,y_0) is a critical point of v

Similarly, if (x_0,y_0) is a critical point of v, then (x_0,y_0) is a critical point of u.

(b) Let (x_0,y_0) be a critical point of u.

Then, as in part (a), $u_x(x_0,y_0) = 0$ and $v_x(x_0,y_0) = 0$

Thus $f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0) = 0$

$\Rightarrow z_0$ is a critical point of f.

(c) If $z_0 = x_0 + iy_0$ is a critical point of f, then

$$f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0) = 0 \implies u_x(x_0,y_0) = v_x(x_0,y_0) = 0$$

By the Cauchy-Riemann equations,

$$u_y(x_0,y_0) = -v_x(x_0,y_0) = 0 \quad \text{and} \quad v_y(x_0,y_0) = u_x(x_0,y_0) = 0$$

$\implies (x_0,y_0)$ is a critical point of u and v

4.) (a) Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{5+4\sin t}\,dt$

Let $z(t) = e^{it}$. Then $z'(t) = iz$ and $\sin t = \dfrac{z - z^{-1}}{2i}$

Thus

$$\int_0^{2\pi} \frac{1}{5+4\sin t}\,dt = \int_C \frac{1}{5+4\left(\frac{z+z^{-1}}{2}\right)} \cdot \frac{1}{iz}\,dz$$

where C is the positively oriented unit circle.

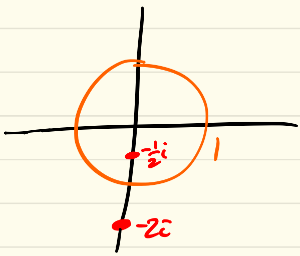
Now $$\int_C \frac{1}{5+4\left(\frac{z-z^{-1}}{2i}\right)} \cdot \frac{1}{iz}\,dz = \int_C \frac{1}{2z^2+5iz-2}\,dz$$

Since $2z^2+5iz-2 = 0 \iff z = \dfrac{-5i \pm \left((5i)^2 - 4(2)(-2)\right)^{1/2}}{2(2)} = \dfrac{-5i \pm 3i}{4} = -2i, -\frac{1}{2}i$

we can factor $2z^2+5iz-2 = 2(z+2i)(z+\frac{1}{2}i)$

Notice, $-1/2\,i$ is enclosed by C and $-2i$ is not enclosed by C.

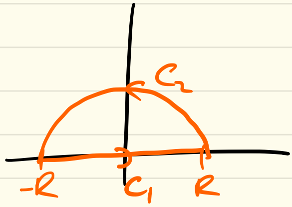
Thus $f(z) = \dfrac{1}{2(z+2i)}$ is analytic on and inside of C.



By the Cauchy Integral Formula,

$$\int_C \frac{1}{2z^2+5iz-2}\,dz = \int_C \frac{\frac{1}{2(z+2i)}}{2z+i}\,dz = 2\pi i f(-\tfrac{1}{2}i) = \frac{2\pi}{3}.$$

4.) Let $C_R = C_1 \cup C_2$, where C_1 is the line segment from $-R$ to R in the real axis and C_2 is the upper half of the circle centered at 0 of radius R oriented positively.



Then
$$\int_C \frac{1}{z^2+2z+2}\,dz = \int_{C_1} \frac{1}{z^2+2z+2}\,dz + \int_{C_2} \frac{1}{z^2+2z+2}\,dz$$
$$= \int_{-R}^{R} \frac{1}{x^2+2x+2}\,dx + \int_{C_2} \frac{1}{z^2+2z+2}\,dz$$

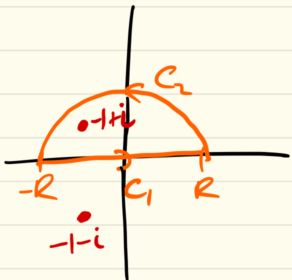
$$\Rightarrow \int_{-R}^{R} \frac{1}{x^2+2x+2}\,dx = \int_C \frac{1}{z^2+2z+2}\,dz - \int_{C_2} \frac{1}{z^2+2z+2}\,dz$$

Evaluate $\int_C \frac{1}{z^2+2z+2}\,dz$:

Since $x^2+2x+2=0 \iff x = \frac{-2 \pm \sqrt{4-4(2)}}{2} = -1 \pm i$

we have $\frac{1}{x^2+2x+2} = \frac{1}{(x-(-1+i))(x-(-1-i))}$

When R is large, $-1+i$ is enclosed by C and $-1-i$ is not enclosed by C_R.



Let $f(z) = \frac{1}{z-(-1-i)}$. Then f is analytic on and inside C, so by the Cauchy Integral Formula,
$$\int_C \frac{1}{z^2+2z+2}\,dz = \int_C \frac{\frac{1}{z-(-1-i)}}{z-(-1+i)}\,dz = 2\pi i f(-1+i) = \pi$$

Evaluate $\int_{C_2} \frac{1}{z^2+2z+2}\,dz$ as $R \to \infty$

Since $|z^2+2z+2| \geq \left||z|^2 - 2|z| - 2\right| = |R^2-2R-2| = R^2-2R-2$ for large R

we have $\left|\frac{1}{z^2+2z+2}\right| \leq \frac{1}{R^2-2R-2}$

Thus $\left|\int_{C_2} \frac{1}{z^2+2z+2}\,dz\right| \leq \frac{\text{Length}(C_2)}{R^2-2R-2} = \frac{\pi R}{R^2-2R-2}$

Since $\lim_{R\to\infty} \frac{\pi R}{R^2-2R-2} = 0$, we have $\lim_{R\to\infty}\left|\int_{C_2} \frac{1}{z^2+2z+2}\,dz\right| = 0$

$\Rightarrow \lim_{R\to\infty} \int_{C_2} \frac{1}{z^2+2z+2}\,dz = 0$

Thus $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+2x+2}\,dx$

$$= \lim_{R\to\infty}\left(\int_{C} \frac{1}{z^2+2z+2}\,dz + \int_{C_2} \frac{1}{z^2+2z+2}\,dz\right)$$

$$= \pi$$