Homework 9 Solutions

1) $z_n = \left(\frac{2\sqrt{2}}{3-3i}\right)^n$

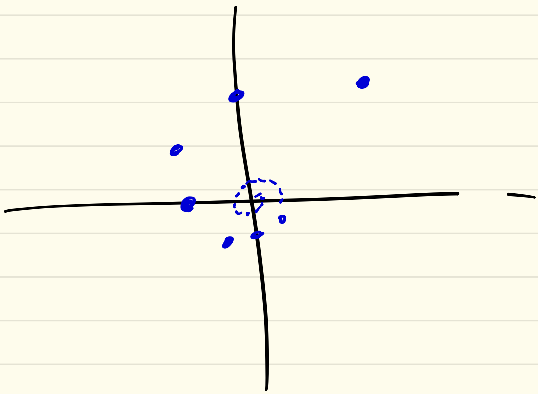
a) $\frac{2\sqrt{2}}{3-3i} = \frac{2\sqrt{2}}{3}\frac{1}{1-i}\cdot\frac{1+i}{1+i} = \frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3}i = \frac{2}{3}e^{i\frac{\pi}{4}}$

$z_1 = \frac{2}{3}\cdot e^{i\frac{\pi}{4}}$

$z_2 = \frac{4}{9}e^{i\frac{\pi}{2}}$

$z_3 = \frac{8}{27}e^{i\frac{3\pi}{4}}$

$\vdots$



b) By part (a), $z_n = \left(\frac{2}{3}\right)^n e^{i\frac{n\pi}{4}}$. Since $e^{\frac{in\pi}{4}}$ is on the unit circle for all n and $\frac{2}{3}<1$,

$\lim_{n\to\infty} z_n = 0.$

Thus the sequence converges to 0.

2) Let $z_n = x_n + iy_n$. Then $|z_n| = \sqrt{x_n^2 + y_n^2}$

If $\lim_{n\to\infty} z_n = 0$, then $0 = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i\lim_{n\to\infty} y_n$

$\Rightarrow \lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$

$\Rightarrow \lim_{n\to\infty} |z_n| = \lim_{n\to\infty}\sqrt{x_n + y_n} = \sqrt{\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n} = 0$

since the square root function is continuous

3.) Let S_N be the N^{th} partial sum of $\sum_{n=1}^{\infty} z_n$.
then $\lim_{N\to\infty} S_N = S = \sum_{n=1}^{\infty} z_n$

Moreover, the N^{th} partial sum of $\sum_{n=1}^{\infty} \overline{z_n}$ is

$$\overline{z}_1 + \overline{z}_2 + \dots + \overline{z}_N = \overline{z_1 + \dots + z_N} = \overline{S_N}$$

Thus $\lim_{N\to\infty} \overline{S_N} = \overline{S}$ and so $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$

4.) a) $\sum_{n=1}^{\infty} z^n$ converges when $|z|<1$ and diverges when $|z| \geq 1$

When $|z|<1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

Thus $\sum_{n=1}^{\infty} z^n = \left(\sum_{n=0}^{\infty} z^n\right) - 1 = \frac{1}{1-z} - 1 = \frac{z}{1-z}$

b) Let $z = re^{i\theta}$, $0<r<1$. Since $\sum_{n=1}^{\infty} z^n$ is convergent,

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} \left(r^n\cos(n\theta) + i r^n \sin(n\theta)\right)$$

$$= \sum_{n=1}^{\infty} r^n \cos(n\theta) + i \sum_{n=1}^{\infty} r^n \cos(n\theta)$$

c) $\sum_{n=1}^{\infty} r^n \cos(n\theta) + i \sum_{n=1}^{\infty} r^n \cos(n\theta) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$

$$= \frac{z}{1-z} \cdot \frac{\overline{1-z}}{\overline{1-z}} = \frac{z}{1-z} \cdot \frac{1-\bar{z}}{1-\bar{z}} = \frac{z - |z|^2}{1 - 2(z+\bar{z}) + |z|^2}$$

$$= \frac{r\cos\theta + ir\sin\theta - r^2}{1 - 2r\cos\theta + r^2} = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} + i \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} \;\&\; \sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

5.) (a) Notice, $\sum_{n=1}^{\infty}\left(\frac{3}{1-i}\right)^n$ is a geometric series.

Since $\left|\frac{3}{1-i}\right| = \frac{|3|}{|1-i|} = \frac{3}{\sqrt{2}} > 1$, $\sum_{n=1}^{\infty}\left(\frac{3}{1-i}\right)^n$ diverges

Alternatively, the divergence test can be used.

(b) $\sum_{n=1}^{\infty}\left|\frac{e^{\frac{\pi}{5}ni}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the Calc II p-series test

thus $\sum_{n=1}^{\infty}\frac{e^{\frac{\pi}{5}ni}}{n^2}$ is absolutely convergent and

thus $\sum_{n=1}^{\infty}\frac{e^{\frac{\pi}{2}ni}}{n^2}$ is convergent.

(c) We'll use the Dirichlet test.

Let $a_n = \frac{1}{n}$ and $b_n = e^{\frac{\pi}{2}ni}$

then $a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$.

Consider the N^{th} partial sums of $\sum_{n=1}^{\infty} b_n$

$B_1 = e^{\frac{\pi}{2}i} = i$, $B_2 = e^{\frac{\pi}{2}i} + e^{\pi i} = i-1$, $B_3 = e^{\frac{\pi}{2}i} + e^{\pi i} + e^{\frac{3\pi}{2}i} = i-1-i = -1$

$B_4 = e^{\frac{\pi}{2}i} + e^{\pi i} + e^{\frac{3\pi}{2}i} + e^{2\pi i} = i-1-i+1 = 0$

$B_5 = i$, $B_6 = i-1$, $B_7 = -1$, $B_8 = 0$, ...

Thus B_N is either $i, i-1, -1,$ or 0 for all N.

$\Rightarrow |B_N| = \left|\sum_{n=1}^{N} b_n\right| \leq |i-1| = \sqrt{2}$ for all N

Thus by the Dirichlet Test, $\sum_{n=1}^{\infty}\frac{e^{\frac{\pi}{2}ni}}{n}$ is convergent.

(d) Since $\lim\limits_{n\to\infty} \frac{n^2 - in + 1}{2-(3+i)n^2} = \frac{-1}{3+i} \neq 0$,

$\sum\limits_{n=1}^{\infty} \frac{in^2 - n + 1}{2-(3+i)n^2}$ diverges by the Divergence Test.

6.) $\sum\limits_{n=0}^{\infty} \frac{i}{2^n}(z+2i)^n$

a) $\lim\limits_{n\to\infty} \sqrt[n]{\left|\frac{i}{2^n}(z+2i)^n\right|} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{|z+2i|^n}{2^n}} = \lim\limits_{n\to\infty} \frac{|z+2i|}{2} = \frac{|z+2i|}{2}$

By the root test, the series converges if $\frac{|z+2i|}{2} < 1 \Rightarrow |z+2i| < 2$

Thus the radius of convergence is 2 and the disk of convergence is $|z+2i| < 2$

b) On the boundary of the disk of convergence, $|z+2i|=2$.

Thus $\lim_{n\to\infty} \left| \frac{i(z+2i)^n}{2^n} \right| = \lim_{n\to\infty} \frac{|z+2i|^n}{2^n} = \lim_{n\to\infty} 1 = 1 \neq 0$

Thus $\lim_{n\to\infty} \frac{i(z+2i)^n}{2^n} \neq 0$, by problem 2.

$\Rightarrow \sum_{n=0}^{\infty} \frac{i}{2^n}(z+2i)^n$ diverges when z is on the boundary, by the divergence test.

7.) $\sum_{n=0}^{\infty} \frac{1}{(n+1)3^n} z^n$

a) $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+2)3^{n+1}} z^{n+1}}{\frac{1}{(n+1)3^n} z^n} \right| = \lim_{n\to\infty} \frac{n+1}{3(n+2)} |z| = \frac{1}{3}|z|$

By the ratio test, the series converges if $\frac{1}{3}|z| < 1$
$\Rightarrow |z| < 3$

Thus the radius of convergence is 3 and the disk of convergence is $|z| < 3$

b) Let z be on the boundary, so $|z|=3$ and suppose $z \neq 3$.

Let $a_n = \frac{1}{n}$ and $b_n = \frac{z^n}{3^n} = \left(\frac{z}{3}\right)^n$

Then $a_{n+1} > a_n$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

The N^{th} partial sum of $\sum_{n=1}^{\infty} b_n$ is

$$B_N = 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots + \left(\frac{z}{3}\right)^N$$

$$= \left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots + \left(\frac{z}{3}\right)^N\right]\left(\frac{1-\frac{z}{3}}{1-\frac{z}{3}}\right) = \frac{1-\left(\frac{z}{3}\right)^{N+1}}{1-\frac{z}{3}}$$

$$\Rightarrow |B_N| = \left|\frac{1-\left(\frac{z}{3}\right)^{N+1}}{1-\frac{z}{3}}\right| \leq \frac{1+\left(\frac{|z|}{3}\right)^{N+1}}{\left|1-\frac{z}{3}\right|} = \frac{2}{\left|1-\frac{z}{3}\right|} \text{ for all } N$$

Thus, by the Dirichlet Test, the series converges.

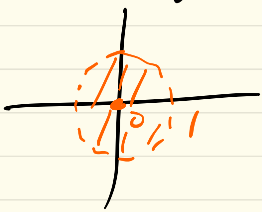
If $z=3$, then $\sum_{n=0}^{\infty} \frac{1}{(n+1)3^n} z^n = \sum_{n=0}^{\infty} \frac{1}{(n+1)}$,

which diverges by the Calc II p-series test.

8.) $\sum_{n=1}^{\infty} \frac{i}{n^2} z^n$

a) $\lim_{n\to\infty} \left| \dfrac{\frac{i}{(n+1)^2} z^{n+1}}{\frac{i}{n^2} z^n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} |z| = |z|$

By the ratio test, the series converges if $|z|<1$ and diverges if $|z|>1$.
Thus the radius of convergence is 1 and the disk of convergence is $|z|<1$



b) On $|z|=1$, $\sum_{n=1}^{\infty} \left| \frac{i}{n^2} z^n \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges by the p-series test.

$\Longrightarrow \sum_{n=1}^{\infty} \frac{i}{n^2} z^n$ is absolutely convergent and thus convergent.