Exam 1 Solutions

1. Let $z = 1 - \sqrt{3}i$.

   (a) Evaluate z^5 and express your answer in rectangular form.

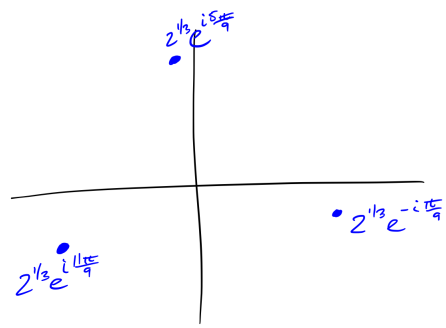
   Since $|z| = 2$ and $\text{Arg}z = -\frac{\pi}{3}$, in exponential form, $z = 2e^{-i\frac{\pi}{3}}$.
   Thus $z^5 = 32e^{-i\frac{5\pi}{3}} = 32(\cos(-\frac{5\pi}{3}) + i\sin(-\frac{5\pi}{3})) = 16 + 16\sqrt{3}i$.

   (b) Evaluate $\text{Log}z$ and express your answer in rectangular form.

   $\text{Log}(1 - \sqrt{3}i) = \ln 2 - \frac{\pi}{3}i$.

   (c) Find the 3^{rd} roots of z. Plot and label them in the complex plane.

   The 3rd roots are $2^{\frac{1}{3}}e^{-i\frac{\pi}{9}}, 2^{\frac{1}{3}}e^{i\frac{5\pi}{9}}$, and $2^{\frac{1}{3}}e^{i\frac{11\pi}{9}}$.



2. The goal of this problem is to show that if $\sin z = 0$, then z is a real number of the form $z = n\pi$, where $n \in \mathbb{Z}$.

   (a) Let $\sin z = 0$ and set $w = e^{iz}$. Use the definition of $\sin z$ to show that $w^2 = 1$.

   $$0 = \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{w - w^{-1}}{2i} \Rightarrow w - \frac{1}{w} = 0 \Rightarrow w^2 = 1$$

   (b) Use part (a) to show that $z = n\pi$, where $n \in \mathbb{Z}$.

   Since $w^2 = 1$, we have that $(e^{iz})^2 = e^{2iz} = 1$. Thus $2z = 2n\pi$, where $n \in \mathbb{Z}$, and so $z = n\pi$, where $n \in \mathbb{Z}$.

3. Let z be a complex number such that e^z is on the imaginary axis. What are the possible values of the real and imaginary parts of z?

   Let $z = x + iy$. Then $e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$. Since e^z is on the imaginary axis, $e^x \cos y = 0$ and so $\cos y = 0$. Thus $Imz = y = \frac{(2n+1)}{2}\pi$, where $n \in \mathbb{Z}$, and $Rez = x \in \mathbb{R}$.

4. (a) Show that $\lim_{z\to 0} \frac{|z|}{z}$ does not exist.

   Let $z = x + iy$. Then $\frac{|z|}{z} = \frac{\sqrt{x^2+y^2}}{x+iy}$.

   As $z \to 0$ along the positive real axis, $\frac{|z|}{z} = \frac{|x|}{x} = 1 \to 1$. As $z \to 0$ along the negative real axis, $\frac{|z|}{z} = \frac{|x|}{x} = -1 \to -1$. Thus the limit does not exist.

   (b) Use part (a) to show that $f(z) = |z|$ is not differentiable at 0.

   By definition, $f'(0) = \lim_{\Delta z\to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{|\Delta z|}{\Delta z}$.
   By part (a), we know that this limit does not exist. Thus f is not differentiable at the origin.

5. Determine whether the following statements are True or False. If true, no explanation is needed. If false, update the statement so that it is true.

   (a) $\text{Arg}(\frac{z_1}{z_2}) = \text{Arg}z_1 - \text{Arg}z_2$ for all $z_1, z_2 \in \mathbb{C}$.

   False, $\arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$ for all $z_1, z_2 \in \mathbb{C}$

   (b) The unit sphere can be identified with the complex plane $\mathbb{C}$.

   False, the unit sphere can be identitfied with $\mathbb{C} \cup \{\infty\}$

   (c) $\text{Log}z$ is continuous on its domain.

   False, $\text{Log}z$ is not continuous at points on the negative real axis.

   (d) The image of the imaginary axis under the function e^z is the unit circle.

   True