Exam 2 Solutions

1. Let $f(z) = (\overline{z})^2$.
   (a) Express f in the form $f(z) = u(x, y) + iv(x, y)$.

   Solution: $f(z) = (\overline{z})^2 = (x - iy)^2 = (x^2 - y^2) - 2xyi$.

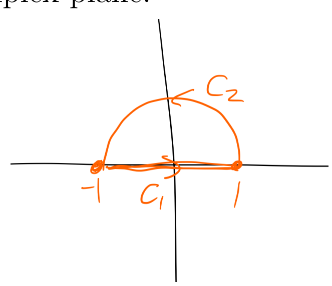
   (b) Determine where $f'(z)$ exists and find its value. Explain your reasoning.

   Solution: Since $u_x = 2x$, $u_y = -2y$, $v_x = -2y$, and $v_y = -2x$, the Cauchy-Riemann equations are satisfied if and only if $2x = -2x$ and $2y = -2y$, or $x = 0$ and $y = 0$. Thus if $z \neq 0$, $f'(z)$ does not exist. Since the Cauchy-Riemann equations are satisfied at $z = 0$ and since the partial derivatives of u and v exist in an ϵ-neighborhood of $z = 0$ and are continuous at $z = 0$, $f'(0)$ exists. Moreover, $f'(0) = u_x(0, 0) + iv_x(0, 0) = 0$.

   (c) Determine where f is analytic. Explain your reasoning.

   Solution: Since $f'(z)$ does not exist when $z \neq 0$, f is not analytic at all points $z \neq 0$. f is also not analytic at $z = 0$, since $f'(z)$ does not exist in every ϵ-neighborhood of $z = 0$. So, f is not analytic anywhere.

2. Let C_1 be the straight line from -1 to 1 and let C_2 be the upper half of the unit circle from 1 to -1. Let $C = C_1 \cup C_2$.
   (a) Sketch C_1 and C_2 in the complex plane.



   (b) Parametrize C_1 and C_2. Be sure to give bounds on your parameters.

   Solution: We can parametrize C_1 by $z_1(t) = t$, $-1 \leq t \leq 1$ and C_2 by $z_2(t) = e^{it}$, $0 \leq t \leq \pi$.

   (c) Compute $\displaystyle\int_C f(z)\, dz$, where $f(z) = \overline{z}$.

   Solution:

$$\int_{C_1} \overline{z}\, dz = \int_{-1}^{1} t\, dt = 0$$

$$\int_{C_2} \overline{z}\, dz = \int_0^{\pi} e^{-it} i e^{it}\, dt = \int_0^{\pi} i\, dt = i\pi$$

Thus $\displaystyle\int_C f(z)\, dz = 0 + i\pi = i\pi$

3. Let C be an arbitrary curve from $z = i\sqrt{\pi}$ to $z = 0$ and let $f(z) = z \sin(z^2)$. Compute $\displaystyle\int_C f(z)\, dz$.

   Solution: Since $f(z)$ has antiderivative $F(z) = -\frac{1}{2}\cos(z^2)$ on all of $\mathbb{C}$, by the fundamental theorem of contour integrals,

   $$\int_C z \sin(z^2)\, dz = -\frac{1}{2}\cos(z^2)\Big|_{i\sqrt{\pi}}^{0} = -1$$

4. Suppose f is entire and $\overline{f(z)} = f(z)$ for all $z \in \mathbb{C}$. Show that f is constant on $\mathbb{C}$.

   Solution: Let $f(z) = u(x,y) + iv(x,y)$. Then since $\overline{f(z)} = f(z)$, we have that $u(x,y) - iv(x,y) = u(x,y) + iv(x,y)$ and so $v(x,y) = 0$. Since f is entire, the Cauchy-Riemann equations are satisfied at all points. Thus $u_x = v_y = 0$ and $u_y = -v_x = 0$ at all points. Therefore, $f'(z) = u_x + iv_x = 0$ for all $z \in \mathbb{C}$. Thus f is constant on $\mathbb{C}$.

5. Fill in the blanks.
   If $f'(z_0)$ exists, then $u_\theta =$ $\underline{\ -rv_r\ }$ and $v_\theta =$ $\underline{\ ru_r\ }$ at z_0.

6. Determine whether the following statements are True or False. If true, no explanation is needed. If false, update the statement in a **nontrivial** way so that it is true.

   (a) $f(z) = \log z$ $(r > 0, -\frac{3\pi}{4} < \theta \leq \frac{5\pi}{4})$ is a branch of $\log z$.

   Solution: False. $\log z$ is not analytic (or even continuous) on the region described and so f is not a branch.

   (b) If f is continuous on $\mathbb{C}$, then $\displaystyle\int_{-C} f(z)\, dz = \int_C f(z)\, dz$.

   Solution: False. $\displaystyle\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$

   (c) If $f'(z_0)$ exists, then $u_\theta = -rv_r$ and $v_\theta = ru_r$ at z_0.

   True. These are the Cauchy-Riemann equations.