Math 421 Final Solutions

1. Determine *all* singular points of the following functions (except $\infty$).
Classify the isolated singular points and calculate the residue at each one (except $\infty$).

   (a) $f(z) = \dfrac{(z^6-1)e^{\frac{1}{z}}}{z^4}$

   **Solution**: $f$ has an isolated singularity at $z=0$. It's Laurent series at $z=0$ is

$$\frac{(z^6-1)e^{\frac{1}{z}}}{z^4} = \left(z^2 - \frac{1}{z^4}\right)\sum_{n=0}^{\infty}\frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!z^{n-2}} - \sum_{n=0}^{\infty}\frac{1}{n!z^{n+4}}$$

   There is one term with an exponent of $-1$, namely the fourth term in the first series, which is $\dfrac{1}{6z}$. Thus $\operatorname*{Res}_{z=0} f(z) = \dfrac{1}{6}$.

   (b) $f(z) = \dfrac{\text{Log}z}{z-1}$

   **Solution**: $f$ has an isolated singularity at $z=1$ and nonisolated singularities at all points on the nonpositive real axis. Since $\text{Log}(1)=0$, we cannot apply the theorem involving poles to deduce that 1 is a pole and to calculate the residue. Instead, we must consider the Laurent series of this function centered at $z=1$.

   Since $\text{Log}z$ is analytic at 1, it has a Taylor series $\text{Log}z = \sum_{n=0}^{\infty} a_n(z-1)^n$ in a disk centered at $z=1$. Thus $f$ has a Laurent series centered at $z=1$ of the form

$$\frac{\text{Log}z}{z-1} = \sum_{n=0}^{\infty} a_n(z-1)^{n-1} = \frac{a_0}{z-1} + a_1 + a_2(z-1) + \cdots$$

   Thus $\operatorname*{Res}_{z=1} h(z) = a_0 = \text{Log}(1) = 0$. Moreover, since there are no negative exponents in the Laurent series, 1 is a removable singularity.

2. Evaluate the following contour integrals using any method we discussed in class.

   (a) $\displaystyle\int_C \csc^2 z\,dz$, where $C$ be the contour given by the graph of the function $y=\cos x$ from $x=\frac{\pi}{2}$ to $x=0$. Express your answer in terms of $e$.

   **Solution**: First notice that $C$ begins at $z=\frac{\pi}{2}$ and ends at $z=i$. Since $f(z)=\csc^2 z$ has an antiderivative of $F(z) = -\cot z$ for all $z \neq n\pi$ for all $n \in \mathbb{Z}$, and $C$ does not contain any of these points, we can apply the Fundamental Theorem of Contour Integrals.

$$\int_C \csc^2 z\,dz = -\cot z\Big|_{\frac{\pi}{2}}^{i} = -\cot i = -\frac{\cos i}{\sin i} = \frac{i(e^2+1)}{e^2-1}$$

(b) $\displaystyle\int_C \frac{z}{(z-2)^2}\,dz$, where $C$ is the positively-oriented circle $|z-i|=2$.

**Solution**: Since $f(z) = \dfrac{z}{(z-2)^2}$ is analytic everywhere except at $z=2$, which is not on or enclosed by $C$, $f$ is analytic on and inside $C$. Thus by the Cauchy-Goursat theorem, $\displaystyle\int_C \frac{z}{(z-2)^2}\,dz = 0$

(c) $\displaystyle\int_C \frac{z^2-1}{2z^3+2z^2+z}\,dz$, where $C$ is the positively-oriented unit circle.

**Solution**: We first find the singularities of $f(z) = \dfrac{z^2-1}{2z^3+2z^2+z}$. By factoring and using the quadratic formula, we have:

$$2z^3+2z^2+z = 0 \Rightarrow z(2z^2+2z+1) = 0 \Rightarrow z = 0, -\frac{1}{2}+\frac{1}{2}i, -\frac{1}{2}-\frac{1}{2}i$$

Since all of the singularities are enclosed by $C$, we will compute the integral by computing the residue at infinity.

$$g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1-z^2}{z(z^2+2z+2)}$$

Let $\phi(z) = \dfrac{1-z^2}{2z^2+z+2}$. Then $\phi$ is analytic and nonzero at $z=0$. Thus $z=0$ is a pole of order 1 and $\operatorname*{Res}_{z=0} g(z) = \phi(0) = \frac{1}{2}$. Thus $\operatorname*{Res}_{z=\infty} f(z) = -\frac{1}{2}$ and

$$\int_C \frac{z^2-1}{2z^3+2z^2+z}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = \pi i$$

3. Consider the function $f(z) = \dfrac{\cos^2(\pi z) - 2e^{\sin(\pi z)}}{\sin(\pi z)}$.

(a) Show that $z=n$ is a singular point of $f$ for all $n \in \mathbb{Z}$ and that

$$\operatorname*{Res}_{z=n} f(z) = \begin{cases} \frac{1}{\pi} & \text{if } n \text{ is odd} \\ -\frac{1}{\pi} & \text{if } n \text{ is even} \end{cases}$$

**Solution**: $f$ has an isolated singularity at $z=n$ for all $n \in \mathbb{Z}$. Let $p(z) = \cos^2(\pi z) - 2e^{\sin(\pi z)}$ and $q(z) = \sin(\pi z)$. Since $q(n) = 0$ and $q'(n) = \pi\cos n\pi \neq 0$, $q$ has a zero of order 1 at $z=n$ for all $n \in \mathbb{Z}$. Moreover, notice that $q'(n) = \pi \cos n\pi = \pi$ if $n$ is even and $q'(n) = \pi\cos n\pi = -\pi$ if $n$ is odd. Now, since $p$ and $q$ are analytic at $z=n$ and $p(n) = -1 \neq 0$, $f$ has a pole of order 1 at $z=n$ for all $n$ and

$$\operatorname*{Res}_{z=n} f(z) = \frac{p(n)}{q'(n)} = \begin{cases} \frac{1}{\pi} & \text{if } n \text{ is odd} \\ -\frac{1}{\pi} & \text{if } n \text{ is even} \end{cases}$$

(b) Let $C_{\frac{2k+1}{2}}$ be the positively-oriented circle $|z| = \frac{2k+1}{2}$, where $k$ is a nonnegative integer. Show that

$$\int_{C_{\frac{2k+1}{2}}} f(z)\,dz = \begin{cases} 2i & \text{if } k \text{ is odd} \\ -2i & \text{if } k \text{ is even} \end{cases}$$

**Solution**: $C_{\frac{2k+1}{2}}$ encloses the singular points $-k, \ldots, -1, 0, 1, \ldots, k$. Thus we can use the Cauchy Residue Theorem to calculate the contour integral:

$$\int_{C_{\frac{2k-1}{2}}} f(z) = f(z)\,dz = 2\pi i(\operatorname*{Res}_{z=-k} f(z) + \cdots + \operatorname*{Res}_{z=0} f(z) + \cdots + \operatorname*{Res}_{z=k} f(z))$$

By part (a), if $k$ is odd, then the sum of the above residues is $-\frac{1}{\pi}$ and if $k$ is even, then the sum is $\frac{1}{\pi}$. Thus we have

$$\int_{C_{\frac{2k-1}{2}}} f(z)\,dz = \begin{cases} 2i & \text{if } k \text{ is odd} \\ -2i & \text{if } k \text{ is even} \end{cases}$$

4. Suppose $f$ is analytic at a point $z_0$. Show that there exists a positive real number $R$ such that if $C_r$ is a circle of radius $r$ centered at $z_0$ and $r < R$, then $\displaystyle\int_{C_r} f(z)\,dz = 0$.

**Solution**: By definition, since $f$ is analytic at $z_0$, it is analytic in an $\epsilon-$neighborhood $U$ of $z_0$. Thus, if $r < \epsilon$, then the circle $C_r$ given by $|z - z_0| = r$ is contained in $U$. Thus by the Cauchy-Goursat Theorem, the contour integral of $f$ along $C_r$ is 0.

5. Suppose $f$ is analytic everywhere except at $z_1, z_2 \in \mathbb{C}$. Let $C_1$ and $C_2$ be positively-oriented circles of radius $r$ centered at $z_1$ and $z_2$, respectively, and let $C$ be a positively-oriented contour enclosing $C_1$ and $C_2$. Show that

$$\left| \int_C f(z)\,dz \right| \leq 2\pi r(M_1 + M_2)$$

where $M_1$ and $M_2$ are the maximum values of $|f(z)|$ on $C_1$ and $C_2$, respectively.

**Solution**: Since $f$ is analytic on each curve and on the region between the curves,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

By the triangle inequality, we have

$$\left| \int_C f(z)\,dz \right| \leq \left| \int_{C_1} f(z)\,dz \right| + \left| \int_{C_2} f(z)\,dz \right|$$

Since $C_1$ and $C_2$ have length $2\pi r$, we have that

$$\left|\int_C f(z)\,dz\right| \le \left|\int_{C_1} f(z)\,dz\right| + \left|\int_{C_2} f(z)\,dz\right| \le 2\pi r M_1 + 2\pi r M_2 = 2\pi r(M_1 + M_2)$$

6. Is it possible for the power series $\displaystyle\sum_{n=1}^{\infty} n^n z^n$ to be the Maclaurin series expansion of some function $f$ that is analytic at 0?

**Solution**: Let's first figure out where this series converges and diverges.

$$\lim_{n\to\infty} \sqrt[n]{|n^n z^n|} = \lim_{n\to\infty} n|z| = \begin{cases} 0 & \text{if } z = 0 \\ \infty & \text{if } z \neq 0 \end{cases}$$

Thus, by the root test, the series diverges for all $z \neq 0$. By Taylor's theorem, if $f$ is analytic at $z = 0$, then $f$ has a (convergent) Maclaurin series expansion in some $\epsilon$-neighborhood of 0. But, since the series diverges in every $\epsilon$-deleted neighborhood of 0, it cannot be the Maclaurin series of some function that is analytic at 0.

7. I saw the following calculation on the internet recently:

$$i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \Rightarrow i^2 = 1 \Rightarrow i = \pm 1$$

There is obviously something wrong with this calculation. What is it?

**Solution**: $\sqrt{-1} = (-1)^{\frac{1}{2}}$ is the set of square roots of $-1$, namely $\sqrt{-1} = \{i, -i\}$. Thus the product $\sqrt{-1}\sqrt{-1}$ is not well-defined and so equating it with 1 does not make sense.