Practice Problems

Disclaimer: This problem set does not include an exhaustive list of the topics that we have covered or the topics that might appear on the final exam. So, do not rely solely on these problems in your preparation for the exam. In addition to working through these problems, I highly recommend going through problems from old homework and exams.

1. Solve the equation $z^4 = 8 - 8\sqrt{3}i$ (i.e. find the 4th roots of $z = 8 - 8\sqrt{3}i$) and express the solutions in rectangular coordinates.

2. Is $f(z) = \cos x - \sin y + iy$ differentiable at $z = \frac{3\pi}{2} - \frac{\pi}{2}i$? Is it analytic there? Explain your reasoning.

3. Does there exist a function $v(x, y)$ such that $f(z) = \cos x \sin y + iv(x, y)$ is entire? If so, find such a v. If not, explain why. Does $u(x, y) = \cos x \sin y$ have a harmonic conjugate on $\mathbb{C}$? Why or why not?

4. Let $f(z) = e^{2z}$. Does $|f(z)|$ have a maximum value on the region $|z| < 1$? How about on the region $|z| \le 1$? Show that the maximum value of $|f(z)|$ on the unit circle is e^2. Use this value to find an upper bound for $|f^{(3)}(0)|$.

5. Give a list of all bounded, entire functions.

6. Use the fundamental theorem of contour integrals and/or the Cauchy-Goursat theorem to evaluate the following integrals. Be sure to explain why they apply.

   (a) $\displaystyle\int_C (z + 1)\, dz$, where C is the positively-oriented contour formed from the line segments between 0, 1, and i.

   (b) $\displaystyle\int_C z^{-2}\, dz$, where C is any contour from $z = i$ to $z = -i$.

   (c) $\displaystyle\int_C z^{-2}\, dz$, where C is the positively-oriented unit circle.

7. Use the Cauchy integral formula to evaluate the following integrals. Be sure to explain why it applies.

   (a) $\displaystyle\int_C z^{-2}\, dz$, where C is the positively-oriented unit circle.

   (b) $\displaystyle\int_C \frac{\sin z}{(z-1)(z^2+1)}\, dz$, where C is the positively-oriented circle $|z - 1| = 1$

   (c) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2}\, dx$

8. Determine all points at which the following power series converge and diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}(z - i)^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} n!(z - i)^n$

(c) $\sum_{n=0}^{\infty} n(z-i)^n$

9. Classify the isolated singular points of the following functions by constructing Laurent series. Find the residues corresponding to these singular points.

   (a) $f(z) = \dfrac{z+1}{z^3 - z^2}$

   (b) $g(z) = \dfrac{\cosh z - 1}{z^2}$

10. Explain why 0 and ∞ are not isolated singularities of $f(z) = \tan(\frac{1}{z})$.

11. Use the Cauchy residue theorem to evaluate the following integrals.

    (a) $\int_C z^{-2}\, dz$, where C is the positively-oriented unit circle.

    (b) $\int_0^{2\pi} \dfrac{4\cos 2t}{5 - 4\cos t}\, dt$

12. Find the residue at infinity of $f(z) = \dfrac{\cos(\frac{1}{z})}{z^2(z^2-1)}$ and use it to evaluate $\int_C f(z)\, dz$, where C is the positively oriented circle $|z| = 2$.

13. Show that $f(z) = \dfrac{2\cos(z^2)}{1 + z - e^z}$ has a pole of order 2 at $z = 0$.

14. For additional practice, redo old exams and homework and work through problems in the textbook.