Final Practice Problem Solutions

1.) $z = -8 - 8\sqrt{3}i = 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 16e^{-i\frac{2\pi}{3}}$

$$\Rightarrow z^{1/4} = \left\{2e^{-i\frac{\pi}{6}},\ 2e^{i\frac{\pi}{3}},\ 2e^{i\frac{5\pi}{6}},\ 2e^{i\frac{4\pi}{3}}\right\}$$

$$= \left\{2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right), 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right\}$$

$$= \left\{\sqrt{3} - i,\ 1 + \sqrt{3}i,\ -\sqrt{3} + i,\ -1 - \sqrt{3}i\right\}$$

2.) Let $u(x,y) = \cos x - \sin y$, $v(x,y) = y$

then $u_x = -\sin x$ $\quad v_x = 0$. These are continuous
$u_y = \cos y$ $\quad v_y = 1$ everywhere.

$u_x = v_y \iff x = \pm\frac{3\pi}{2}, \pm\frac{7\pi}{2}, \pm\frac{11\pi}{2}, \ldots$
$u_y = -v_x \iff y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Thus f is differentiable at $z = \frac{3\pi}{2} - \frac{\pi}{2}i$

However, it is not analytic there since the Cauchy Riemann equations are not satisfied for all points in $0 < \left|z - \left(\frac{\pi}{2} - \frac{3\pi}{2}i\right)\right| < \frac{\pi}{2}$

3.) Recall that if f is entire, then $u(x,y) = \cos x \sin y$ is harmonic on $\mathbb{C}$. However,

$u_x = -\sin x \sin y \quad u_y = \cos x \cos y \quad \Rightarrow u_{xx} + u_{yy} \neq 0,$
$u_{xx} = -\cos x \sin y \quad u_{yy} = -\cos x \sin y$

$\Rightarrow u$ is not harmonic on $\mathbb{C}$ $\Rightarrow$ there is no function v such that f is entire. In particular, u does not have a harmonic conjugate.

4.) By the Maximum Modulus Principle, since f is analytic and nonconstant on $|z|<1$, f does not have a maximum on $|z|<1$. It does have a max on $|z|\leq 1$, however, which occurs on the boundary $|z|=1$.

Let $z=x+iy$.

$|f(z)|=|e^{2z}|=|e^{2x}e^{2iy}|=e^{2x}|e^{i(2y)}|=e^{2x}$

If $|z|=\sqrt{x^2+y^2}=1$, then the largest x can be is 1.

Thus $|f(z)|=e^{2x}\leq e^2$ for all z on unit circle (since e^x is an increasing function)

By the Cauchy Inequality, since f is analytic on ~~\$~~ inside the unit circle, which is centered at 0,

$$|f^{(3)}(0)| \leq \frac{3!e^2}{1} = 6e^2.$$

5.) By Liouville's theorem, the only entire, bounded functions are constant functions.

These are all of the form $f(z)=c$, where $c\in\mathbb{C}$.

6.) a) Since $f(z)=z+1$ is analytic on and inside C, which is closed, by the Cauchy-Goursat theorem,

$$\int_C z+1 \, dz = 0.$$

b) Since $f(z)=z^{-2}$ has an antiderivative $F(z)=-z^{-1}$ on $\mathbb{C}-\{0\}$ and C is in $\mathbb{C}-\{0\}$, by the fund. thm,

$$\int_C \frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_i^{-i} = \frac{1}{i}+\frac{1}{i} = \frac{2}{i} = -2i$$

c) Since $f(z)=\frac{1}{z^2}$ has an antiderivative on $\mathbb{C}-\{0\}$, and C is closed in $\mathbb{C}-\{0\}$, by the Fundamental Theorem of Contour Integrals,

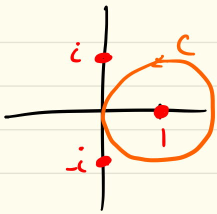
$$\int_C z^{-2}\,dz = 0.$$

2.) a) Let $f(z)=1$. Then f is analytic on & inside C. Thus

$$\int_C \frac{1}{z^2}\,dz = 2\pi i\, f'(0) = 2\pi i\,(0) = 0$$

b) Let $f(z)=\frac{\cos z}{z^2+1}$.

Then f is analytic on & inside C since it is analytic everywhere except $z=\pm i$.



Thus

$$\int_C \frac{\cos z}{(z-1)(z^2+1)}\,dz = \int_C \frac{\frac{\cos z}{z^2+1}}{z-1}\,dz = 2\pi i f(1) = \pi i \cos(1)$$

c) See Solutions to HW8.

8.) a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}(z-i)^n$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}(z-i)^{n+1}}{\frac{2^n}{n!}(z-i)^n}\right| = \lim_{n\to\infty}\frac{2}{n+1}|z-i| = 0 \text{ for all } z$$

Thus, by the ratio test, the series converges for all $z\in\mathbb{C}$.

b) $\sum_{n=0}^{\infty} n!(z-i)^n$

$$\lim_{n\to\infty}\left|\frac{(n+1)!(z-i)^{n+1}}{n!(z-i)^n}\right| = \lim_{n\to\infty}(n+1)|z-i| = \begin{cases}\infty & \text{if } z\neq i\\ 0 & \text{if } z=i\end{cases}$$

Thus, by the ratio test, the series converges if and only if $z=i$.

c) $\sum_{n=0}^{\infty} n(z-i)^n$

$$\lim_{n\to\infty}\left|\frac{(n+1)(z-i)^{n+1}}{n(z-i)^n}\right| = \lim_{n\to\infty}\frac{n+1}{n}|z-i| = |z-i|$$

By the ratio test, the series converges when $|z-i|<1$ and diverges when $|z-i|>1$.
When $|z-i|=1$, notice that

$$\lim_{n\to\infty}|n(z-i)^n| = \lim_{n\to\infty} n|z-i|^n = \lim_{n\to\infty} n = \infty \neq 0$$

thus $\lim_{n\to\infty} n(z-i)^n \neq 0$ (by #2 on HW9)

and so the series diverges by the divergence test.

$\Rightarrow$ Series converges on $|z-i|<1$, diverges on $|z-i|\geq 1$.

9.) (a) $f(z) = \frac{z+1}{z^3-z^2}$ has isolated singularities $z=0$ and $z=1$

$z=0$

$$\frac{z+1}{z^3-z^2} = -\frac{z+1}{z^2}\,\frac{1}{1-z} = -\frac{z+1}{z^2}\sum_{n=0}^{\infty} z^n = \left(-\frac{1}{z}-\frac{1}{z^2}\right)\left(1+z+z^2\cdots\right)$$

$$= -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$

$$-\frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 + \cdots$$

$$= -\frac{1}{z^2} - \frac{2}{z} - 2 - 2z^2 - \cdots$$

Thus $z=0$ is a pole of order 2 and $\operatorname{Res}_{z=0} f(z) = -2$

$z=1$

$$\frac{z+1}{z^3-z^2} = \frac{z+1}{z-1}\,\frac{1}{z^2} = \frac{z+1}{(z-1)}\left(\frac{1}{1-(1-z)}\right)^2$$

$$= \frac{(1-z)-2}{1-z}\left(\sum_{n=0}^{\infty}(1-z)^n\right)\left(\sum_{n=0}^{\infty}(1-z)^n\right)$$

$$= \left(1-\frac{2}{1-z}\right)\left(1+(1-z)+(1-z)^2+\cdots\right)\left(1+(1-z)+(1-z)^2+\cdots\right)$$

Multiplying this out, we see that there is only one term with negative exponent: $-\frac{2}{1-z}$

$\Rightarrow z=1$ is a pole of order 1

and $\operatorname{Res}_{z=1} f(z) = -2.$

(b) $g(z)$ has an isolated singularity at $z=0$.

$$g(z)=\frac{\cosh z-1}{z^2}=\frac{1}{z^2}\left(-1+\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}\right)$$

$$=\frac{1}{z^2}\left(-1+\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)\right)$$

$$=\frac{1}{z^2}\left(\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right)$$

$$=\frac{1}{2!}+\frac{z^2}{4!}+\frac{z^4}{6!}+\cdots$$

There are no terms with negative exponents
$\Rightarrow$ 0 is a removable singularity.

10.) $f(z)=\tan\left(\frac{1}{z}\right)=\dfrac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$

f has singularities at $z=0$, $\frac{1}{(2k+1)\pi}$ for all $k\in\mathbb{Z}$.

Now, every ε-deleted nbhd of 0 $(0<|z|<\varepsilon)$
contains a singularity, since for all k such that $(2k+1)\pi>\frac{1}{\varepsilon}$
we have $\frac{1}{(2k+1)\pi}<\varepsilon$.

Thus $z=0$ is not isolated.

Similarly, for every neighborhood U of ∞ $(|z|>N)$,
there exists singularities in U

Thus ∞ is not an isolated singularity.

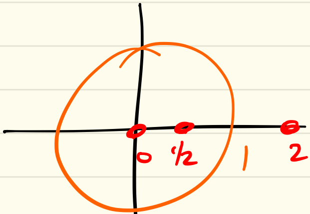
11) a) Let $f(z) = \frac{1}{z^2}$. This is already in Laurent series form, and $\operatorname{Res}_{z=0} f(z) = 0$.

Thus $\int_C \frac{1}{z^2}\,dz = 2\pi i \operatorname{Res}_{z=0} f(z) = 0$.

b) $\int_0^{2\pi} \frac{4(\cos^2 t - \sin^2 t)}{5 - 4\cos t}\,dt = \int_C \frac{4\left(\frac{z+z^{-1}}{2}\right)^2 - 4\left(\frac{z-z^{-1}}{2i}\right)^2}{5 - 4\left(\frac{z+z^{-1}}{2}\right)} \frac{1}{iz}\,dz$

$= 2i \int_C \frac{z^4+1}{z^2(2z-1)(z-2)}\,dz$, where C is the positively oriented unit circle.

$f(z) = \frac{z^4+1}{z^2(2z-1)(z-2)}$ has isolated singular points at $z=0$, $z=\frac{1}{2}$, and $z=2$.



Since 0 & $\frac{1}{2}$ are enclosed by C, by the Cauchy Residue Theorem,

$2i \int_C f(z)\,dz = 2i \left(2\pi i \operatorname{Res}_{z=0} f(z) + 2\pi i \operatorname{Res}_{z=1/2} f(z) \right)$

Residue at $z=0$

Let $\phi(z) = \frac{z^4+1}{(2z-1)(z-2)}$. Then ϕ is analytic an nonzero at $z=0$

Thus $z=0$ is a pole of order 2 and

$\operatorname{Res}_{z=0} f(z) = \frac{\phi'(0)}{1!} = \left.\frac{(4z^3)(2z-1)(z-2) - \left(2(z-2) + (2z-1)\right)(z^4+1)}{\left((2z-1)(z-2)\right)^2}\right|_{z=0} = \frac{5}{4}$

Residue at $z = 1/2$

Let $\phi(z) = \dfrac{z^4+1}{2z^2(z-2)}$. Then ϕ is analytic an nonzero at $z = 1/2$

Thus $z = 1/2$ is a pole of order 1 and

$$\operatorname*{Res}_{z=1/2} f(z) = \phi(1/2) = \frac{-17}{12}$$

Therefore, $\displaystyle\int_0^{2\pi} \frac{4(\cos^2 t - \sin^2 t)}{5-4\cos t}\,dt = 2i\left(2\pi i\left(\frac{5}{4}\right) + 2\pi i\left(\frac{-17}{12}\right)\right)$

$$= -5\pi + \frac{17\pi}{3} = \frac{2\pi}{3}$$

12.) $f(z) = \dfrac{\cos(\frac{1}{z})}{z^2(z^2-1)}$ has isolated singularities

at $0, \pm 1$, which are all enclosed by C.

Thus $\displaystyle\int_C \frac{\cos(\frac{1}{z})}{z^2(z^2-1)}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$

$$g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\cos(z)}{\frac{1}{z^2}\left(\frac{1}{z^2} - \frac{1}{z}\right)}\right) = \frac{z^2\cos(z)}{1-z}$$

Since $\dfrac{z^2\cos z}{1-z}$ is analytic at $z=0$, 0 is a removable singularity of g.

Thus $\operatorname*{Res}_{z=0} g(z) = 0 \implies \operatorname*{Res}_{z=\infty} f(z) = 0$.

$$\implies \int_C \frac{\cos(\frac{1}{z})}{z^2(z^2-1)}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 0$$

13.) Let $p(z) = 2\cos z^2$, $q(z) = 1 + z - e^z$.

Then p and q are analytic at $z=0$ and $p(0) \neq 0$.

Moreover, since $g(0) = 0$, $g'(0) = 0$, $g''(0) = -1 \neq 0$,

g has a zero of order 2 at $z=0$.

Thus $f(z) = \dfrac{2\cos z^2}{1+z-e^z}$ has a pole of order 2 at $z=0$.