Exam 1 Solutions

1. Let

$$f(x,y) = \begin{cases} x + \dfrac{x^4y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Show that f is differentiable at $(0,0)$, given that $\dfrac{\partial f}{\partial x}(0,0) = 1$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$.

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \mathbf{D}f(0,0)\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{\left|\left|\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right|\right|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x + \frac{x^4y^4}{x^2+y^2} - 0 - \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{\left|\left|\begin{bmatrix} x \\ y \end{bmatrix}\right|\right|} = \lim_{(x,y)\to(0,0)} \frac{x + \frac{x^4y^4}{x^2+y^2} - x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^2)^{\frac{3}{2}}}$$

Now, $0 \leq \dfrac{x^4y^4}{(x^2+y^2)^{\frac{3}{2}}} \leq \dfrac{x^4y^4}{(x^2)^{\frac{3}{2}}} = \dfrac{x^4y^4}{|x|^3} = |x|y^4$.

Since $\lim\limits_{(x,y)\to(0,0)} 0 = \lim\limits_{(x,y)\to(0,0)} |x|y^4 = 0$, by the Squeeze Theorem, we have that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^2)^{\frac{3}{2}}} = 0$. Thus f is differentiable at $(0,0)$.

(b) Since f is differentiable at $(0,0)$, the linearization L of f at $(0,0)$ is a good approximation of f near $(0,0)$. Use L to approximate $f(.1,.1)$.

Solution:

$L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) = 0 + 1(x-0) + 0(y-0) = x.$

Thus $f(.1,.1) \approx L(.1,.1) = .1$

2. Suppose $\mathbf{x}_0$ is a nondegenerate critical point of a C^2 function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ and

$$Hf(\mathbf{x}_0) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Is $\mathbf{x}_0$ a local minimum, local maximum, or saddle point?

Solution: Since $\det[-2] = -2 < 0$, $\det \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} = 5 > 0$ and $\det Hf(\mathbf{x}_0) = -5 < 0$, the Hessian is negative-definite and so f has a local maximum at $\mathbf{x}_0$.

(b) What is the index of $\mathbf{x}_0$?

Solution: Since $Hf(\mathbf{x}_0)$ is negative-definite, it has 3 negative eigenvalues. Thus $\mathbf{x}_0$ is an index 3 critical point.

3. Let $f(x, y) = (1 - y)^3 x^2 + y^2$.

(a) Show that $(0, 0)$ is the only critical point of f.

Solution: $\dfrac{\partial f}{\partial x} = 2x(1 - y)^3$ and $\dfrac{\partial f}{\partial y} = -3(1 - y)^2 x^2 + 2y$. Setting $\dfrac{\partial f}{\partial x}$ equal to 0, we have either $x = 0$ or $y = 1$. If $y = 1$, then $\dfrac{\partial f}{\partial y} = 2 \neq 0$ and so there is no critical point with $y = 1$. If $x = 0$, then $\dfrac{\partial f}{\partial y} = 2y$. Setting this equal to zero, we have that $y = 0$. Thus $(0, 0)$ is the only critical point.

(b) Does f have a local minimum, local maximum, or saddle point at $\mathbf{x}_0$.

Solution: The Hessian of f is $Hf(x, y) = \begin{bmatrix} 2(1 - y)^3 & -6x(1 - y)^2 \\ -6x(1 - y)^2 & 6(1 - y)x^2 + 2 \end{bmatrix}$.
Thus $Hf(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Since the eigenvalues of $Hf(0, 0)$ are both 2 (and positive), $Hf(0, 0)$ is positive-definite and thus f has a local minimum at $(0, 0)$.

(c) It is tempting to assume that since $(0, 0)$ is the only critical point and f is continuous on $\mathbb{R}^2$, then f must have an absolute minimum at $(0, 0)$ (This would certainly be true for a function of one variable). Show that this is not the case: f does not have an absolute minimum at $(0, 0)$.

Solution: If $f(0, 0) = 0$ was the absolute minimum of f, then there would be no no function values less than 0. However, for example, $f(3, 2) = -5 < 0$. Thus f does not have an absolute minimum at $(0, 0)$.

4. Find the absolute maximum and minimum function values of $f(x, y) = e^{-(x^2+2y^2)}$ on the region $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. At what points do these extrema occur?

Solution: First, we find the critical points. Since the partial derivatives $\dfrac{\partial f}{\partial x} = -2xe^{-(x^2+2y^2)}$ and $\dfrac{\partial f}{\partial y} = -4ye^{-(x^2+2y^2)}$ are both zero only when $x = 0$ and $y = 0$, $(0, 0)$ is the only critical point of f and $f(0,0) = 1$.

Now, let $g(x,y) = x^2+y^2$. Then $\nabla g = \langle 2x, 2y\rangle$ and $\nabla f = \langle -2xe^{-(x^2+2y^2)}, -4ye^{-(x^2+2y^2)}\rangle$. Thus we must solve the system:

$$-2xe^{-(x^2+2y^2)} = \lambda 2x$$

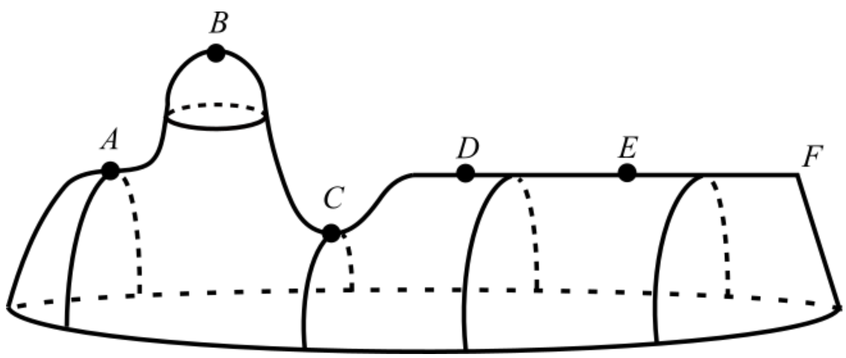
$$-2ye^{-(x^2+2y^2)} = \lambda 2y$$

$$x^2 + y^2 = 1$$

Suppose $x \neq 0$. Then we can divide both sides of the first equation by $2x$ to obtain $\lambda = -e^{-(x^2+2y^2)}$. Plugging this into the second equation, we obtain $-4ye^{-(x^2+2y^2)} = -e^{-(x^2+2y^2)}2y \Rightarrow 4y = 2y \Rightarrow y = 0$. Plugging this into the third equation, we have $x = \pm 1$. If, on the other hand, $x = 0$, then by the third equation, $y = \pm 1$. Plugging these points into f, we have $f(\pm 1, 0) = e^{-1}$ and $f(0, \pm 1) = e^{-2}$.

Thus the absolute maximum is 1, which occurs at $(0,0)$ and the absolute minimum is e^{-2}, which occurs at $(0,1)$ and $(0,-1)$.

5. The graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is depicted below.



Match the critical points labelled A, B, C, D, E, F with the appropriate descriptions below. More than one point can be put in the same blank space. Not every blank space needs to be filled. (No work is necessary for this problem)

Index 2, nondegenerate critical point(s): B

Index 1, nondegenerate critical point(s): C

Index 0, nondegenerate critical point(s): ____________

Isolated, degenerate critical point(s): A

Nonisolated, degenerate critical point(s): ___D and E___

Critical point(s) at which f is not differentiable: _____F_____