Exam 2 Solutions

1. Let $W \subset \mathbb{R}^3$ be the solid bounded by $z = 1 - y^2$, $z = 0$, $x = 0$, and $x = 1$.

   (a) Find the volume of W.

   Solution:
   By considering the sketch of W above, we can describe W by
   $W = \{(x, y, z) \mid 0 \le x \le 1, -1 \le y \le 1, 0 \le z \le 1 - y^2\}$. Thus the volume of W is

   $$\iiint_W 1\, dV = \int_0^1 \int_{-1}^1 \int 0^{1-y^2} 1\, dz\, dy\, dx = \int_0^1 \int_{-1}^1 (1 - y^2)\, dy\, dx = \int_0^1 \frac{4}{3}\, dx = \frac{4}{3}$$

   (b) Suppose W has constant density C. Find the mass of W.

   Solution: The mass of W is $\iiint_W C\, dV = C \iiint_W 1\, dV = \frac{4}{3}C$, by part (a).

2. Let D be the region in the first quadrant bounded by $xy = 1$, $xy = 2$, $y = x$, and $y = 2x$. Use the change of variables $u = x$, $v = xy$ to rewrite the double integral $\iint_D xe^{xy}\, dA$ in terms of u and v. Do **not** evaluate the integral.

   Solution: Since $u = x$, $v = xy$, we have that $y = \frac{v}{x} = \frac{v}{u}$. Note that, in the region we are considering, since $x, y > 0$, we have $u, v > 0$. Let $T(u, v) = (u, \frac{v}{u})$. Then $|\det \mathbf{D}T(u, v)| = \left| \det \begin{bmatrix} 1 & 0 \\ -\frac{v}{u}2 & \frac{1}{u} \end{bmatrix} \right| = |\frac{1}{u}| = \frac{1}{u}$, since $u > 0$ in the region we are considering.

   Now, $xy = 1 \Rightarrow v = 1, xy = 2 \Rightarrow v = 2, y = x \Rightarrow v = u^2$, and $y = 2x \Rightarrow v = 2u^2$. Thus the region in the $(u, v)-$plane is bounded by $v = 1, v = 2, v = u^2$, and $v = 2u^2$, where $u, v > 0$. This region is shown below.

   Thus the integral is $\iint_D xe^{xy}\, dA = \int_1^2 \int_{u^2}^{2u^2} ue^v \frac{1}{u}\, du\, dv = \int_1^2 \int_{u^2}^{2u^2} e^v\, du\, dv$

3. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid 1 \le x^2 + y^2 + z^2 \le 2, x \ge 0, y \ge 0\}$.

   (a) Rewrite the triple integral $\iiint_W (x^2 + z^2)e^{x^2+y^2+z^2}\, dV$ in spherical coordinates. Do **not** evaluate the integral.

Solution: W is a quarter of the region between two spheres as depicted below. In spherical coordinates, $W = \{(\rho, \theta, \phi) \mid 1 \le \rho \le \sqrt{2}, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \pi\}$. Thus

$$\iiint_W (x^2+z^2)e^{x^2+y^2+z^2}\, dV = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_1^{\sqrt{2}} (\rho^2\sin^2\phi\cos^2\theta+\rho^2\cos^2\phi)e^{\rho^2}\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

(b) If $e^{x^2+y^2+z^2}$ is the density of W at point (x, y, z), what does the integral in part (a) represent?

Solution: The integral represents the moment of inertia of W about the y-axis.

4. It turns out that the only curves in $\mathbb{R}^3$ that have constant curvature are straight lines, circles, and spirals/helixes. Show that the helix $C \subset \mathbb{R}^3$ parametrized by $\mathbf{c}(t) = (\cos t, \sin t, t)$ has constant curvature.

Solution:

$\mathbf{c}'(t) = (-\sin t, \cos t, 1)$ and $\mathbf{c}''(t) = (-\cos t, -\sin t, 0)$.

$\mathbf{c}'(t) \times \mathbf{c}''(t) = (\sin t, -\cos t, \sin^2 t + \cos^2 t) = (\sin t, -\cos t, 1)$.

Thus $\kappa(t) = \dfrac{||\mathbf{c}'(t) \times \mathbf{c}''(t)||}{||\mathbf{c}'(t)||^3} = \dfrac{\sqrt{\sin^2 t + \cos^2 t + 1}}{(\sqrt{\cos^2 t + \sin^2 t + 1})^3} = \dfrac{\sqrt{2}}{\sqrt{2}^3} = \dfrac{1}{2}$.

Since $\kappa(t) = \frac{1}{2}$ for all t, C has constant curvature.

5. Let $C \subset \mathbb{R}^2$ be parametrized by $\mathbf{c}(t) = (\frac{3}{5}t + 1, 3 - \frac{4}{5}t)$.

(a) Is C smooth? Why or why not?

Solution: Since $\mathbf{c}'(t) = (\frac{3}{5}, -\frac{4}{5})$ is continuous and not equal to $\mathbf{0}$, C is smooth.

(b) Show that $\mathbf{c}$ is a parametrization with respect to arc length.

Solution: Since $||\mathbf{c}'(t)|| = \sqrt{(\frac{3}{5})^2 + (-\frac{4}{5})^2} = 1$ for all t, $\mathbf{c}$ is a parametrization with respect to arc length.

(c) Find the length of C between $(1, 3)$ and $(4, -1)$. (There are multiple ways to do this).

Solution: $(1, 3)$ corresponds to $t = 0$ and $(4, -1)$ corresponds to $t = 5$. Since $\mathbf{c}$ is a parametrization with respect to arc length, the length between these points is 5.

Alternatively, using the arc length formula, we have $\displaystyle\int_0^5 ||\mathbf{c}'(t)||\, dt = \int_0^5 1\, dt = 5$.

Alternatively, since C is a straight line, the arc length is simply the distance between the points $(1, 3)$ and $(4, -1)$, which is $\sqrt{(1-4)^2 + (3-(-1))^2} = 5$.