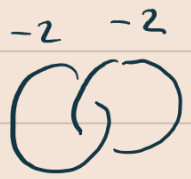
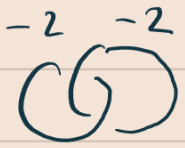
From handle diagrams for 4-manifolds to surgery diagrams for 3 manifolds



X^4



M^3

Link diagrams with each component labelled with an integer have both meanings 4-dimensionally (handle diagram) and 3-dimensionally (surgery diagram).

Q: How are these related?

A: $\partial X = M$ as we'll see

<u>Claim</u>: If A, B are manifolds then $\partial(A \times B) = \partial(A) \times B \cup A \times \partial(B)$.

Ex: $I \times I$ 

Ex: $\partial(B^4) \cong \partial(B^2 \times B^2) = \partial B^2 \times B^2 \cup_\varphi B^2 \times \partial B^2$

$\cong S^3$ $\qquad = S^1 \times B^2 \cup_\varphi B^2 \times S^1$

Note that with care can figure out what φ is.

<u>Claim</u>: If M is a manifold with a decomposition $M = A \cup B$ where A and B are both manifolds then

$\partial M \cong (\partial(A) - \text{gluing region}) \cup (\partial(B) - \text{gluing region})$

Ex:

3-dimensional handle decomp

has boundary

$\partial(0\text{-handle}) - \text{overlap} \cup \partial(1\text{-handle}) - \text{overlap}$

$= S^2 - \text{attaching region} \cup \partial D^2 \times I \cup D^2 \times \partial I - \text{attaching region}$

$= \text{cylinder} \cup_\varphi \text{cylinder}$

$= \text{torus}$

<u>EX</u>:

X

has boundary:

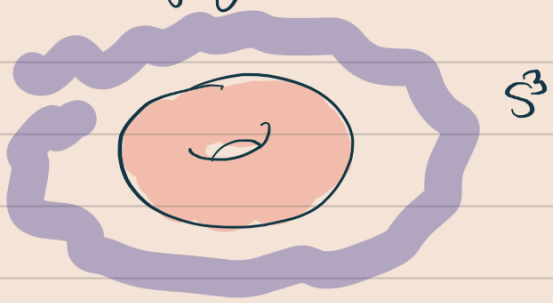
$\partial(0\text{-handle})$ - overlap $\cup$ $\partial(2\text{-handle})$ - overlap

$= S^3 - \text{attaching} \cup S^1 \times B^2 \cup B^2 \times S^1 - \text{attaching}$

$= S^3 - \nu(K) \cup_\varphi B^2 \times S^1$

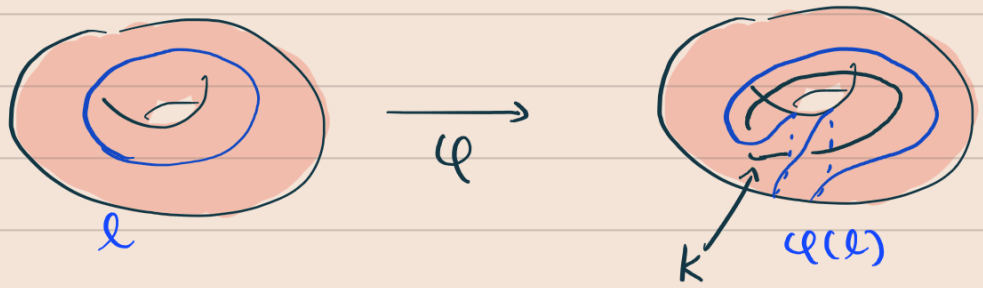
So ∂X is certainly Dehn surgery. But with which framing?

$\partial(2\text{-handle})$



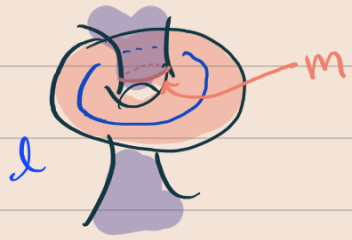
$\partial(0\text{-handle})$

The two orange regions will be identified. How?



φ is determined by where ℓ goes, which is what is encoded by the framing. -2 framing $\Rightarrow$ $lk(\varphi(\ell), K) = -2$

We keep track of the identification, but note that the orange regions are not part of the boundary.



What we need to understand to determine our surgery coefficient is where the meridian of the complementary purple torus goes under φ. But note that the meridian of the purple $B^2 \times S^1$ is isotopic to ℓ. Thus $lk(m, K) = -2$ also. This is equivalent to performing -2 surgery.

We just showed:

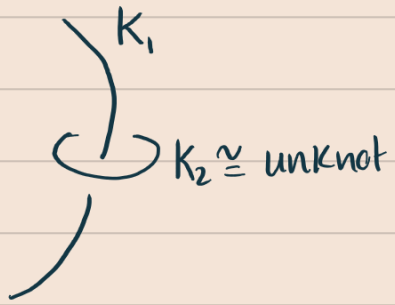
<u>Prop</u>: $$\partial\left(\begin{array}{c}\text{2-handle}\\ \text{with framing } n\\ \text{along } K\end{array} \cup B^4\right) \cong S^3_n(K)$$

Can similarly show that

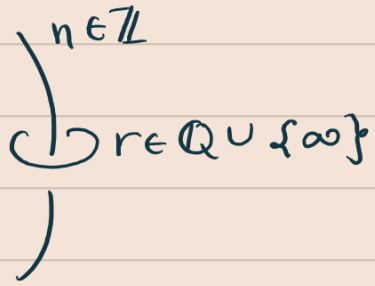
$$\partial\left(\bigcup_{i=1}^{\ell}\left(\begin{array}{c}\text{2-handle}\\ \text{with framing } n_i\\ \text{along } K_i\end{array}\right) \cup B^4\right) \cong S^3_{n_1, n_2, \dots, n_\ell}(K_1, K_2, \dots, K_\ell)$$

<u>Kirby moves</u> | <u>Surgery moves</u>

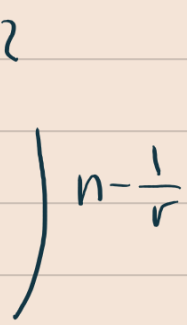
<u>Slam dunk</u>:



with framings:



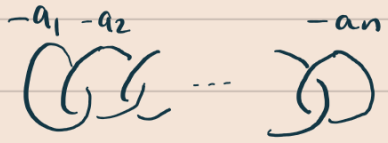
as 3-manifolds



note, this has no 4-D interpretation since coefficients won't be integral here

This gives 2 different surgery descriptions of the same 3-manifold

Lens spaces

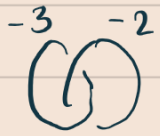
$X^4 =$ 

negative continued fraction

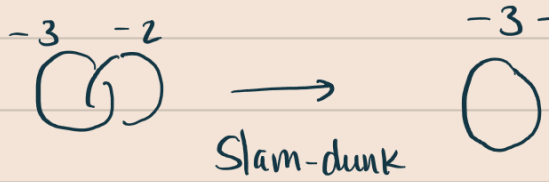
if $[a_1, a_2, \dots, a_n]^- = p/q$

then $\partial X := L(p,q)$

Claim: $L(p,q)$ is $-p/q$ surgery on the unknot $S^3_{-p/q}(U)$

EX: $X =$ 

now thinking of this as a surgery diagram for the boundary

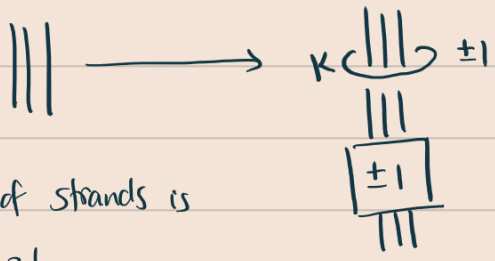


$-3 - \left(\frac{1}{-2}\right) = -5/2$

$= S^3_{-5/2}(U)$

no longer has any 4-D meaning

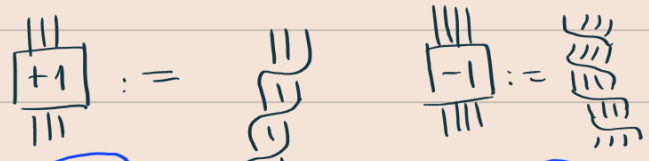
Other moves

Blow up: 

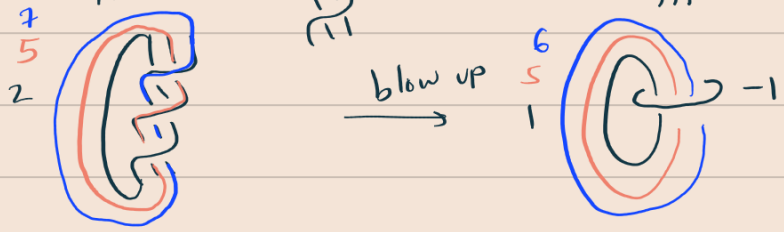
this will usually be 1 for us!

adds $\pm lk(K, K_i)^2$ to framing of strands passing thru K

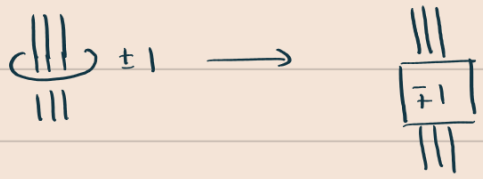
Note: any number of strands is allowed, not just 3!



"a full-twist on K strands"

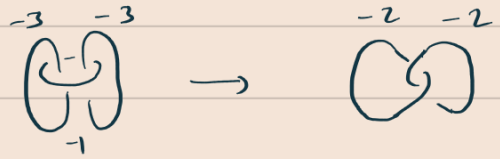
EX: 

blow-down:

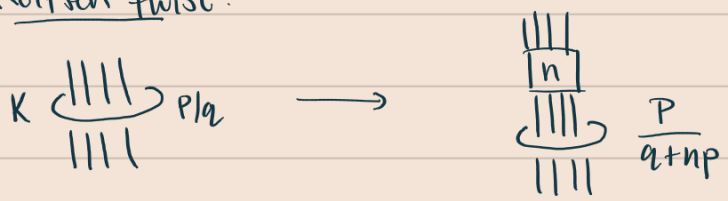


adds $\mp \mathrm{lk}(K, K_i)^2$ to framing if we added ∓ 1 twist box

EX:

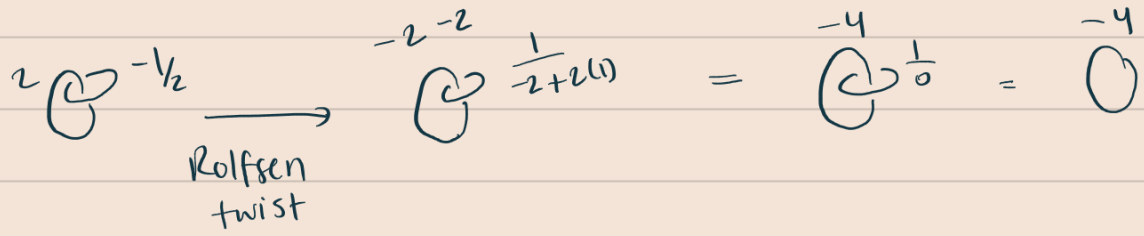


Rolfsen twist:



adds $n(\mathrm{lk}(K_i, K))^2$

EX:



Note: Rolfsen twists include blow-downs as a special case

Why do these moves?

① reduce # of link components (if possible)

② change coefficients which are fractions to be integral or even ± 1

depending on your goal often want either ① or ②

Thm: Suppose L and L' are framed links (in $\mathbb{Q}$) e.g. $S^3(L')$ and $S^3(L)$ determine 3-mfds as surgery. If $S^3(L) \cong S^3(L')$, then

orientation preserving

L and L' can be related by Rolfsen twists (and undoing them), adding or removing components framed by ∞, and isotopy.

- Cool theorem, in principle can principle could produce all surgery diagrams producing a given 3-manifold, but not very useful in practice.