Math 421 Homework 10

1. Find the Taylor series of the following functions. Simplify your answers and be sure to include the disk of convergence.

   (a) $g(z) = \dfrac{4}{3-z}$ centered at $z = 1$.

   (b) $h(z) = \cos(iz + 1)$ centered at $z = i$.

2. Consider $f(z) = \tan z$.

   (a) What is the largest circle within which the Maclaurin series for $f(z) = \tan z$ converges to $\tan z$?

   (b) Write the first two nonzero terms of the Maclaurin series.

3. Without explicitly calculating all of the derivatives, show that if $f(z) = ze^{z^2}$, then $f^{(2n+1)}(0) = (2n+1)2n(2n-1)(2n-2)\cdots(n+1)$ and $f^{(2n)}(0) = 0$ for all $n \geq 1$.

4. Find Laurent series expansions of the following functions centered at $z = 0$. Simplify your answers and include the region on which each series converges.

   (a) $f(z) = \dfrac{1}{z^2}\cosh z$

   (b) $g(z) = \dfrac{4}{3z^3 + 4z^2}$ (Hint: There are two regions to consider here)

   (c) $h(z) = \sin(\dfrac{1}{z^3})$

5. For each function in the previous question, indicate whether $z = 0$ is an essential singularity, a removable singularity, or a pole of order m (find m, in this case). Explain your reasoning.

6. Recall that $\text{Log}z$ ($r > 0$, $-\pi < \theta \leq \pi$) has a singularity at $z = 0$.

   (a) Why can't we use Laurent's theorem to deduce that $\text{Log}z$ has a Laurent series expansion in some annular neighborhood centered at 0?

   (b) Does there exist a singularity z_0 such that $\text{Log}z$ has a Laurent series expansion in some annular neighborhood centered at z_0? If so, around which singularities does $\text{Log}z$ have such an expansion? If not, explain why.

   (c) Does there exist a point $z_0 \in \mathbb{C}$ such that $\text{Log}z$ has a Taylor series expansion in a disk centered at z_0? If so, around which points does $\text{Log}z$ have such an expansion? Carefully explain your reasoning.