Math 421 Homework 11

1. Use Laurent series to calculate the residue at $z = 0$ of each of the following functions and use the residues to calculate the contour integral of each function over the positively-oriented unit circle.

   (a) $f(z) = \dfrac{1}{z}\cosh z$

   (b) $g(z) = \dfrac{4}{3z^3 + 4z^2}$

   (c) $h(z) = \sin(\dfrac{1}{z^3})$

   You may use your Laurent series calculations from the last homework. (Beware, part (a) is a slightly different function from the last homework)

2. Let $f(z) = \dfrac{z+1}{z^2 - 2z}$ and let C be the positively-oriented circle $|z| = 3$.

   (a) Use Laurent series to calculate the residue of each isolated singular point enclosed by C.

   (b) Use the Cauchy Residue Theorem to calculate $\int_C f(z)\,dz$.

   (c) Calculate the residue at infinity of f by considering the Laurent expansion of $\frac{1}{z^2}f(\frac{1}{z})$.

   (d) Show that the sum of the residues (including the residue at infinity) is 0.

   (e) Suppose $g(z)$ is any function with finitely many isolated singular points. Show that the sum of the residues of g (including the residue at infinity) is 0.

3. For each of the following functions, show that every singular point is a pole, without using Laurent series. Determine the order m of each pole and find the corresponding residue (simplify your answers to $a + bi$ form). Thoroughly explain your reasoning.

   (a) $f(z) = \dfrac{\sin z}{(2z - \pi)^3}$

   (b) $g(z) = \dfrac{z+3}{z^2(z^2+2)}$

4. Let C be the positively oriented circle $|z| = 3$ and let $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$. Compute $\int_C f(z)\,dz$. Thoroughly explain your reasoning.

5. Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree n (i.e. $a_n \neq 0$) and let $Q(z) = b_m z^m + \cdots + b_1 z + b_0$ be a polynomial of degree m (i.e. $b_m \neq 0$), where $m \geq n + 2$. Let C be any positively-oriented simple closed curve enclosing all of the zeros of Q.
   Show that $\int_C \dfrac{P(z)}{Q(z)}\,dz = 0$. (Hint: compute the residue at infinity).