Math 425 (Sections 1 and 3) Homework 1 Solutions

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

   (a) Compute the Jacobian $\mathbf{D}f(x,y,z)$.

   Solution: Notice that since $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - z \\ x + y \end{bmatrix}$, we can write f using function notation, namely $f(x,y,z) = (2x - z, x + y)$. Thus the Jacobian is $\mathbf{D}f(x,y,z) = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

   (b) Show that f (as defined above) is differentiable at any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ by using the limit definition of differentiability.

   Solution: Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$. Then

$$\lim_{x \to x_0} \frac{||f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)||}{||\mathbf{x} - \mathbf{x}_0||} = \lim_{x \to x_0} \frac{\left|\left| \begin{bmatrix} 2x - z \\ x + y \end{bmatrix} - \begin{bmatrix} 2x_0 - z_0 \\ x_0 + y_0 \end{bmatrix} - \begin{bmatrix} 2(x - x_0) - (z - z_0) \\ (x - x_0) + (y - y_0) \end{bmatrix} \right|\right|}{||\mathbf{x} - \mathbf{x}_0||}$$

   $= \lim_{x \to x_0} \dfrac{\left|\left| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right|\right|}{||\mathbf{x} - \mathbf{x}_0||} = \lim_{x \to x_0} 0 = 0$. Thus f is differentiable at all points.

   (c) A quicker way to show differentiability is to appeal to Theorem 9 in Section 2.3, which roughly states that if the partial derivatives of f exists at a point and they are continuous in a neighborhood of the point, then f is differentiable at the point. Apply this theorem to show f is differentiable at all points $(x_0, y_0, z_0) \in \mathbb{R}^3$.

   Solution: Since the partial derivatives exist and are continous at all points in $\mathbb{R}^3$, f is automatically differentiable at all points, by the theorem.

2. Let $f(x,y) = -\sqrt[3]{xy}$. In class, we studied the graph of f and observed that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist and that f is not differentiable at $(0,0)$. In this problem, you will concretely show these facts.

   (a) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist. (Hint: use the limit definition of partial derivatives)

   Solution: $\dfrac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{0}{h} = \lim_{h \to 0} 0 = 0.$

   Similarly, $\dfrac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = 0.$

(b) Show that f is not differentiable at $(0,0)$ by using the definition of differentiability.

Solution: To show f is not differentiable, we must show the following limit is not 0.

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \frac{\partial f}{\partial x}(0,0)(x-0) - \frac{\partial f}{\partial y}(0,0)(y-0)}{||(x,y)-(0,0)||} = \lim_{(x,y)\to(0,0)} \frac{-\sqrt[3]{xy}}{\sqrt{x^2+y^2}}.$$

As $(x,y) \to (0,0)$ along the line $y = x$, we have that $\dfrac{-\sqrt[3]{xy}}{\sqrt{x^2+y^2}} = \dfrac{x^{\frac{2}{3}}}{\sqrt{2}|x|} = \dfrac{1}{\sqrt{2}|x^{\frac{1}{3}}|} \to \infty$. Thus the limit does not exist and so f is not differentiable at $(0,0)$.

(c) Once again recall Theorem 9 in Section 2.3 of the textbook, which states that if $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist and the partial derivatives are continuous in a neighborhood of $(0,0)$, then f must be differentiable at $(0,0)$. What can you conclude about the partial derivatives of f in light of this theorem and your findings in parts (a) and (b)?

Solution: Since the partial derivatives exist at $(0,0)$, but f is not differentiable at $(0,0)$, the partial derivatives cannot be continuous in a neighborhood of the origin.

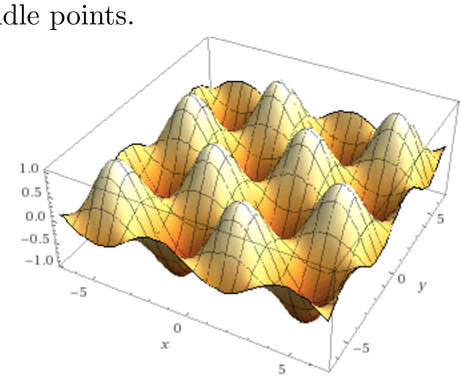
3. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$ that is C^4 but not C^5. Explain your reasoning.

Solution: Let $f(x) = x^{\frac{9}{2}}$. Then $f^{(4)}(x) = \frac{945}{16}x^{\frac{1}{2}}$, which is continuous on $\mathbb{R}$, but $f^{(5)}(x) = \frac{945}{32}x^{-\frac{1}{2}}$, which is not continuous at $x = 0$.

4. Let $g(x,y) = \sin x \sin y$.

(a) Graph g on `https://www.wolframalpha.com`. You'll notice that g looks like an infinite egg carton with infinitely many critical points. Based on the graph, what types of critical points (maxima, minima, saddles) does g have?

Solution: Based on the graph obtained from wolframalpha.com shown below, g has local maxima, minima, and saddle points.



(b) Find one critical point of each type that you found in part (a). Show your work.

Solution: Setting the partial derivative $\frac{\partial g}{\partial x} = \cos x \sin y$ equal to zero, we have that either $x = \frac{(2n+1)\pi}{2}$ or $y = n\pi$, where n is any integer. Similarly, setting $\frac{\partial g}{\partial y} = \sin x \cos y$ equal to zero and solving, we have either $x = n\pi$ or $y = \frac{(2n+1)\pi}{2}$. Thus the critical points are of the form $(n\pi, m\pi)$ and $(\frac{(2n+1)\pi}{2}, \frac{(2m+1)\pi}{2})$, where n and m are any integers.
Consider the critical points $(0,0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, and $(\frac{3\pi}{2}, \frac{\pi}{2})$. The Hessian of g is

$$Hg(x,y) = \begin{bmatrix} -\sin x \sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{bmatrix}.$$

At $(0,0)$, we have

$$\det Hg(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0.$$

Thus f has a saddle point at $(0,0)$. At $(\frac{\pi}{2}, \frac{\pi}{2})$, we have that

$$\det Hg(\frac{\pi}{2}, \frac{\pi}{2}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(\frac{\pi}{2}, \frac{\pi}{2}) = -1 < 0.$$

Thus f has a local maximum at $(\frac{\pi}{2}, \frac{\pi}{2})$. Finally, at $(\frac{3\pi}{2}, \frac{\pi}{2})$, we have that

$$\det Hg(\frac{3\pi}{2}, \frac{\pi}{2}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(\frac{3\pi}{2}, \frac{\pi}{2}) = 1 > 0.$$

Thus f has a local minimum at $(\frac{3\pi}{2}, \frac{\pi}{2})$.

5. Let $h(x,y) = x^4 - y^2$. Show that h has a degenerate critical point at $(0,0)$. In class, we discussed how to determine the concavity in the x- and y-directions around a degenerate critical point. What can you say about the concavity of h in the x- and y-directions around the point $(0,0)$? Explain your reasoning. Graph $h(x,y) = x^4 - y^2$ on `https://wolframalpha.com` to graphically check your work (this last part is for your benefit. You do not have to submit this graph).

   Solution: Since $\frac{\partial^2 h}{\partial x^2}(x,0) = 12x^2 \geq 0$, h is concave up in the x-direction. Since $\frac{\partial^2 h}{\partial y^2}(0,y) = -2 < 0$, h is concave down in the y-direction.

6. Let $f(x,y,z) = x^2 + \frac{1}{3}y^3 + z^2 + 2yz$. Find and classify the critical points of f. What is the index of each critical point?

   Solution: $\dfrac{\partial f}{\partial x} = 2x$, $\dfrac{\partial f}{\partial y} = 2y^2 + 2z$, $\dfrac{\partial f}{\partial z} = 2z + 2y$. Setting these equal to 0, the first equation gives $x = 0$ and the others give $y^2 = -2z$ and $z = -y$. Combining these, we have $y^2 = 2y \Rightarrow y(y-2) = 0 \Rightarrow y = 0, 2$. The corresponding z-values are $0, -2$. Thus the critical points are $(0,0,0)$ and $(0,2,-2)$.

   The Hessian is $H(x,y,z) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4y & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

At the critical points, we have $H(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ and $H(0,2,-2) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

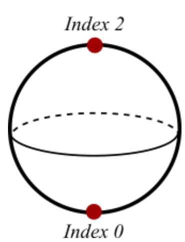
To classify and determine the index of the critical points, we can just find the eigenvalues of the above matrices.

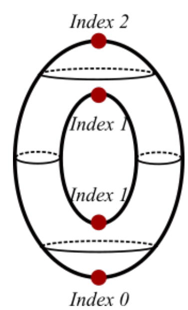
First, let's find the eigenvalues of $H(0,0,0)$:

$\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & 2-\lambda \end{bmatrix} = (2-\lambda)[-\lambda(2-\lambda)-4)] = 0 \Rightarrow \lambda = 2$ or $\lambda^2 - 2\lambda - 4 = 0$. Using the quadratic formula to solve the second equation gives $\lambda = 1 \pm \sqrt{5}$. Now since two of the eigenvalues are positive and one is negative, $H(0,0,0)$ is neither postive- nor negative-definite and so f has a saddle point at $(0,0,0)$. Moreover, $(0,0,0)$ is an index 1 critical point.

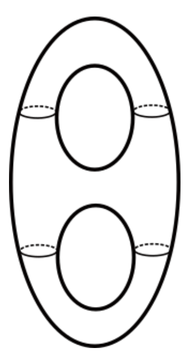
Similarly, the eigenvalues of $H(0,2,-2)$ are 2 and $5 \pm \sqrt{13}$. Since these are all positive, the Hessian is positive-definite and so f has a local minimum at $(0,0,0)$. Moreover, $(0,2,-2)$ is an index 0 critical point.

7. In class we saw that a sphere can be built out of one index 0 critical point and one index 1 critical point and a torus (which is the surface of a donut) can be built out of one index 0 critical point, two index 1 critical points, and one index 2 critical point, as shown below.



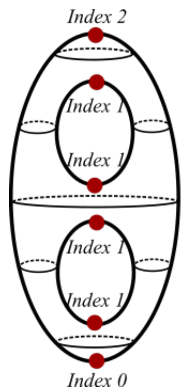


The *genus* of a surface $\mathbb{R}^3$ with no boundary is the number of "donut holes" in the surface. So, the sphere is a surface of genus 0 and the torus a surface of genus 1. Let Σ_g be a surface of genus g. Following the example of the torus we did in class, how many critical points can Σ_2 (which is shown below) be built out of? What are the indices of these critical points?



Let's generalize. How many critical points can Σ_g be built out of? What are the indices of these critical points? This is a conceptual problem and does not require the use of any functions.

Solution: Σ_2 has one index 0 critical point, four index 1 critical points, and one index 2 critical points, as shown below.



More generally, Σ_g has one index 0 critical point, $2g$ index 1 critical points, and one index 2 critical point.