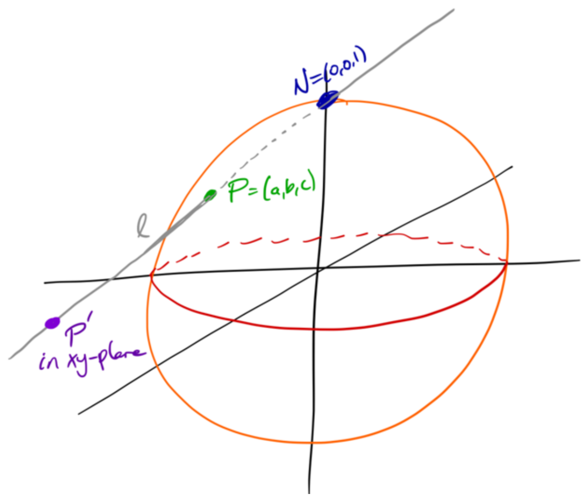
Math 421 Homework 2 Additional Problem

1. Recall the concept of *stereographic projection*. Let S denote the unit sphere in $\mathbb{R}^3$, which is given by $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$. We saw in class that the extended complex plane $\mathbb{C} \cup \{\infty\}$ can be identified with S. The goal of this problem is to prove this fact by deriving a formula for this identification.

   (a) Let $P = (a,b,c)$ be a point on S and let l be the straight line passing through P and $N = (0,0,1)$. Find parametric equations describing the line l (Hint: look back at your Calculus III notes).

   (b) Show that l intersects the xy-plane in exactly one point P'. What is P'? (Your answer should be in terms of a, b and c.)



   (c) Let f be the function from $S - \{(0,0,1)\}$ to the $xy-$plane satisfying $f(P) = P'$. Show that

$$f(x,y,z) = \Big(-\frac{x}{z-1}, -\frac{y}{z-1}, 0\Big)$$

   (d) We can similarly define a function from the $xy-$plane to $S - \{(0,0,1)\}$. This function is

$$g(x,y,0) = \Big(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\Big).$$

   Show that f and g are inverses. That is, show that if (x,y,z) is in S, then $(g \circ f)(x,y,z) = (x,y,z)$ and if $(x,y,0)$ is in the $xy-$plane, then $(f \circ g)(x,y,0) = (x,y,0)$. This implies that f is a one-to-one correspondence between $S - \{(0,0,1)\}$ and the xy-plane, which can be viewed as the complex plane $\mathbb{C}$.

   (e) Now let

$$\phi(x,y,z) = \begin{cases} f(x,y,z) & (x,y,z) \neq (0,0,1) \\ \infty & (x,y,z) = (0,0,1) \end{cases}$$

   This map is a one-to-one correspondence between the entire sphere S and the extended complex plane $\mathbb{C} \cup \{\infty\}$. It is called stereographic projection. (There is nothing to do for this part)