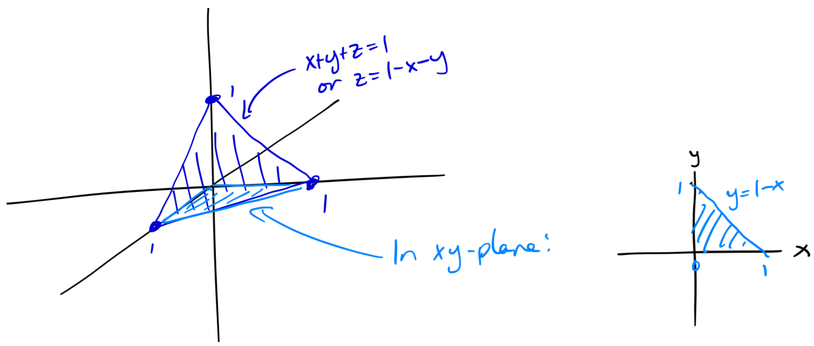
Math 425 (Sections 1 and 3) Homework 2 Solutions

1. Let $f(x, y, z) = x$ and let W be the region in $\mathbb{R}^3$ bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$, and $z = 0$.

   (a) Describe W using set notation:

   $$W = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$$

   Solution:



   $$W = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$$

   (b) Evaluate $\iiint_W f(x, y, z)\, dV$.

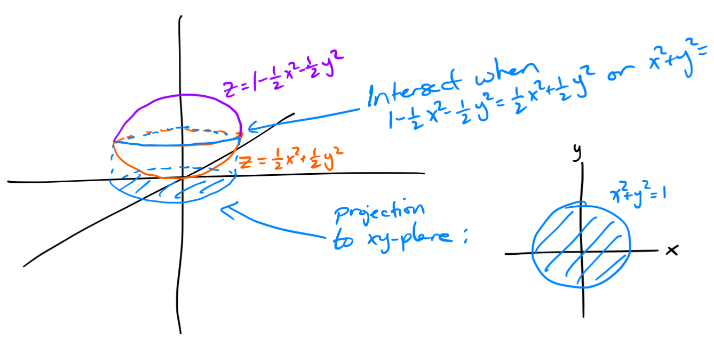
   Solution:

   $$\iiint_W f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} x - x^2 - xy\, dy\, dx$$

   $$= \int_0^1 x(1 - x) - x^2(1 - x) - \frac{1}{2}x(1 - x)^2\, dx = \frac{1}{24}$$

2. Use a triple integral and cylindrical coordinates to find the volume of the flying sauce S bounded by $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$ and $z = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$. To see what it looks like, graph the functions on `geogebra.com` for fun.

   Solution:



First, in cylindrical coordinates, the surfaces are $z = \frac{1}{2}r^2$ and $z = 1 - \frac{1}{2}r^2$. They intersect when $1 - \frac{1}{2}r^2 = \frac{1}{2}r^2$, or when $r = 1$. Thus the region, in cylindrical coordinates is given by $W = \{r, \theta, z) \in \mathbb{R}^3 \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, \frac{1}{2}r^2 \le z \le 1 - \frac{1}{2}r^2$. Thus

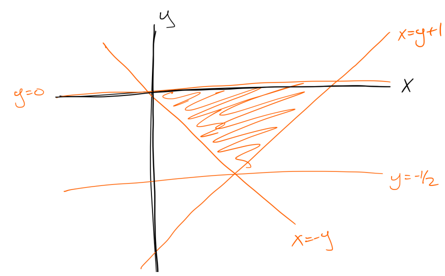
$$Vol(S) = \iiint\limits_S 1dV = \int_0^{2\pi} \int_0^1 \int_{\frac{1}{2}r^2}^{1-\frac{1}{2}r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 r - r^3\,dr\,d\theta = \frac{\pi}{2}$$

3. Evaluate the integral $\int_{-\frac{1}{2}}^0 \int_{-y}^{y+1} (x+y)\cos(\pi(x-y))\,dx\,dy$ using a suitable change of coordinates.

Solution: Let $u = x + y$ and $v = x - y$. Then $x = u - y \Rightarrow v = u - 2y \Rightarrow y = \frac{1}{2}u - \frac{1}{2}v \Rightarrow x = u - (\frac{1}{2}u - \frac{1}{2}v) = \frac{1}{2}u + \frac{1}{2}v$. Let $T(u, v) = (\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v)$.

Then $|\det \mathbf{D}T(u, v)| = \left| \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \right| = \frac{1}{2}$.

Now let's figure out the new bounds. First we can sketch the region in $(x, y)-$coordinates:



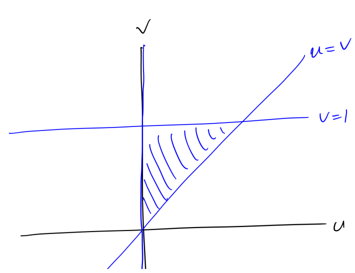
Notice that the boundary of the region is made up of the lines $y = 0$, $x = -y$ and $x = y + 1$. Thus we need to decide what happens to these lines when we change variables.

$y = 0 \Rightarrow \frac{1}{2}u - \frac{1}{2}v = 0 \Rightarrow v = u$

$x = -y \Rightarrow \frac{1}{2}u + \frac{1}{2}v = -\frac{1}{2}u + \frac{1}{2}v \Rightarrow u = 0$

$x = y + 1 \Rightarrow \frac{1}{2}u + \frac{1}{2}v = \frac{1}{2}u - \frac{1}{2}v + 1 \Rightarrow v = 1$

Now let's sketch the region in the $(u, v)-$plane bounded by the lines $v = u, u = 0$ and $v = 1$.



Based on the sketch, we have that $0 \le v \le 1$ and $0 \le u \le v$.

Thus $\int_{-\frac{1}{2}}^0 \int_{-y}^{y+1} (x + y)\cos(\pi(x - y))\,dx\,dy = \int_0^1 \int_0^v u\cos(\pi v)\frac{1}{2}\,dv\,du$

$$= \int_0^1 \frac{u}{2\pi} \sin(\pi u)\, du = \frac{1}{2\pi^2} \text{ (the last equality requires integration by parts).}$$

Note that we can also view this region as $0 \le u \le 1$ and $u \le v \le 1$ and compute the integral with respect to du first. This would take two integration by parts to do.

4. Prove the spherical coordinates formula. More precisely, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$, $W \subset \mathbb{R}^3$, and f is a C^1 function defined on $T(W)$, show that

$$\iiint_{T(W)} f(x, y, z)\, dx\, dy\, dz = \iiint_W f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Solution:

$$\Big|\det \mathbf{D}T(\rho, \theta, \phi)\Big| = \left|\det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}\right|$$

$$= \left|\cos\phi \det \begin{bmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{bmatrix} - \rho\sin\phi \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix}\right|$$

$$= |\cos\phi(-\rho^2 \sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta) - \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)|$$

$$= |-\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi| = |-\rho^2\sin\phi| = \rho^2\sin\phi$$

The last equality is true because $\sin\phi \ge 0$ for all $0 \le \phi \le \pi$.

Now, by the change of variables formula, we have that

$$\iiint_{T(W)} f(x, y, z)\, dx\, dy\, dz = \iiint_W f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)|\det \mathbf{T}(\rho, \theta, \phi)|\, d\rho\, d\theta\, d\phi$$

$$= \iiint_W f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

5. Let $W = \{(x, y, z) \mid x^2 + y^2 + z^2 \le a^2 \text{ and } z \ge 0\}$. Suppose the density of W at any point is twice the distance of that point from the origin.

   (a) Find the mass of W.

   Solution: The density function is $\delta(x, y, z) = 2\sqrt{x^2 + y^2 + z^2}$. In spherical coordinates, we have $\delta(\rho, \theta, \phi) = 2\rho$ and $W = \{(\rho, \theta, \phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$. Thus the mass is $\iiint\limits_W \delta(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a 2\rho^3 \sin\phi\,d\rho\,d\theta\,d\phi = \pi a^4$

(b) Find the center of mass of W.

Solution: Once again, using spherical coordinates, we have

$$\iiint\limits_W x\delta(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a 2\rho^4 \cos\theta \sin^2\phi\,d\rho\,d\theta\,d\phi = 0$$

$$\iiint\limits_W y\delta(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a 2\rho^4 \sin\theta \sin^2\phi\,d\rho\,d\theta\,d\phi = 0$$

$$\iiint\limits_W z\delta(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a 2\rho^4 \cos\phi \sin\phi\,d\rho\,d\theta\,d\phi = \frac{2\pi a^5}{5}$$

Thus we have $\overline{x} = 0, \overline{y} = 0, \overline{z} = \frac{2\pi a^5/5}{\pi a^4} = \frac{2a}{5}$ and so the center of mass is $(0, 0, \frac{2a}{5})$.

(c) Find the moment of inertia about the $z-$axis.

Solution:

$$I_z = \iiint\limits_W (x^2+y^2)\delta(x,y,z)dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a 2\rho^5 \sin^3\phi(\cos^2\theta + \sin^2\theta)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a 2\rho^5 \sin^3\phi\,d\rho\,d\theta\,d\phi = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{a^6}{3} \sin^3\phi\,d\theta\,d\phi = \int_0^{\frac{\pi}{2}} \frac{2\pi a^6}{3} \sin^3\phi\,d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\pi a^6}{3} \sin\phi(1-\cos^2\phi)\,d\phi = \int_0^{\frac{\pi}{2}} \frac{2\pi a^6}{3} \sin\phi - \frac{2\pi a^6}{3} \sin\phi\cos^2\phi)\,d\phi = \frac{4\pi a^6}{9}$$

(The final equality involves u-substitution)

6. Read Example 5 in Section 6.2 of the textbook to see how to evaluate an improper integral of two variables. The key take-away is that in order to integrate a function over an unbounded region we must take a limit. In particular, for a function of two variables, we can use polar coordinates and define

$$\iint\limits_{\mathbb{R}^2} f(x,y)dA = \lim_{a\to\infty} \int_0^{2\pi} \int_0^a f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

Similarly, for a function of three variables, we can use spherical coordinates and define

$$\iiint\limits_{\mathbb{R}^3} f(x,y,z)dV = \lim_{a\to\infty} \int_0^{\pi} \int_0^{2\pi} \int_0^a f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

Use this formula to evaluate $\iiint\limits_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$.

Solution:

$$\iiint\limits_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV = \lim_{a\to\infty} \int_0^{\pi} \int_0^{2\pi} \int_0^{a} \rho^2 e^{-\rho^3} \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \lim_{a\to\infty} \int_0^{\pi} \int_0^{2\pi} \frac{\sin\phi}{3} - \frac{\sin\phi}{3} e^{-a^3} \, d\theta \, d\phi = \lim_{a\to\infty} \int_0^{\pi} \frac{2\pi\sin\phi}{3} - \frac{2\pi\sin\phi}{3} e^{-a^3} \, d\phi$$

$$= \lim_{a\to\infty} \left(\frac{4\pi}{3} - \frac{4\pi}{3} e^{-a^3}\right) = \frac{4\pi}{3}$$

7. (NOT TO BE TURNED IN) Here is some additional integration practice from the textbook: Section 5.5 # 3, 11, 17, 19, 21 and Section 6.2 # 1, 5, 7, 15, 23. Feel free to do even more problems from the textbook for more practice.