Math 425 (Sections 1 and 3) Homework 3 Solutions

1. Let C be the graph of $y = \ln(\sec(x))$.
   (a) Parametrize C.

   Solution: $\mathbf{c}(t) = (t, \ln(\sec(t)))$

   (b) Compute the length of C between $(0,0)$ and $(\frac{\pi}{4}, \ln(\sec(\frac{\pi}{4})))$.
   (Hint: Recall that $\int \sec x\, dx = \ln(\sec x + \tan x) + C$)

   Solution: These points correspond to the parameter values $t = 0$ and $t = \frac{\pi}{4}$. Thus the length is

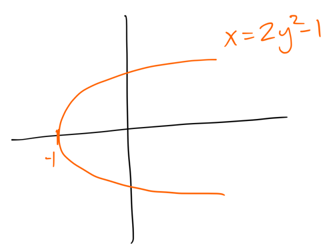
$$\int_0^{\frac{\pi}{4}} ||\mathbf{c}'(t)||\, dt = \int_0^{\frac{\pi}{4}} ||(1, \tan t)||\, dt = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 t}\, dt = \int_0^{\frac{\pi}{4}} \sec t\, dt = \ln(1 + \sqrt{2})$$

2. Let C be parametrized by $\mathbf{c}(t) = (\cos 2t, \cos t)$, where $t \in \mathbb{R}$.
   (a) Compute the tangent vector $\mathbf{c}'(t)$. Can you conclude anything about the smoothness of C? Why or why not?

   Solution: $\mathbf{c}'(t) = (-2\sin 2t, -\sin t)$. Since $\mathbf{c}(0) = (0,0)$, it is possible that C is not smooth. However, we cannot say this for sure, since there might be a parametrization for which the velocity vector is never 0.

   (b) Use the fact that $\cos 2t = 2\cos^2 t - 1$ to find an equation for C involving x and y. Sketch C in the xy-plane.

   Solution: Since $x = \cos 2t = 2\cos^2 t - 1$ and $y = \cos t$, we have that $x = 2y^2 - 1$.



   (c) Based on the equation you found in part (b), give a simpler parametrization of C.

   Solution: Let $y = t$ and $x = 2t^2 - 1$. Thus $\mathbf{c}(t) = (2t^2 - 1, t)$.

   (d) What does the new parametrization tell you about the smoothness of C?

   Solution: Since $\mathbf{c}'(t) = (4t, 1)$ is continuous and $(4t, 1) \neq (0,0)$ for all t, C is smooth.

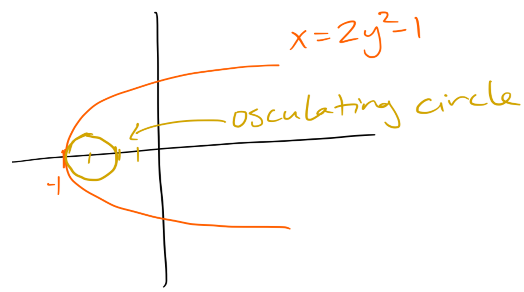
(e) Find the curvature of C at $(-1, 0)$.

Solution: To do this, we can use either parametrization for C. I will use the one found in part (c). At $(-1, 0)$, we have $t = 0$. Viewing C as a curve in $\mathbb{R}^3$, $\mathbf{c}(t) = (2t^2 - 1, t, 0)$. Since $\mathbf{c}'(t) = (4t, 1, 0)$ and $\mathbf{c}''(t) = (4, 0, 0)$ we have that $\mathbf{c}'(0) = (0, 1, 0)$ and $\mathbf{c}''(0) = (4, 0, 0)$. Thus

$$\kappa(0) = \frac{||\mathbf{c}'(0) \times \mathbf{c}''(0)||}{||\mathbf{c}'(0)||^3} = 4$$

(f) Find the equation of the osculating circle of C at $(-1, 0)$. Re-sketch the graph of C along with this osculating circle.

Solution: By definition, the osculating circle passes through $(-1, 0)$, lies on the concave side of C, has curvature equal to the curvature of C at $(-1, 0)$, namely $\kappa(0) = 4$, and is tangent to the tangent vector $\mathbf{c}'(0)$. Moreover, the radius of the circle is $\frac{1}{\kappa(0)} = \frac{1}{4}$. Since $\mathbf{c}'(0) = (0, 1)$, the tangent vector of C at $(-1, 0)$ is vertical. Thus the osculating circle is tangent to the line $x = -1$. Since the radius is $\frac{1}{4}$, the circle has center $(-\frac{3}{4}, 0)$. Thus it has equation $(x + \frac{3}{4})^2 + y^2 = \frac{1}{16}$.



3. Let C be parametrized by $\mathbf{c}(t) = (e^t \cos t, 2, e^t \sin t)$, where $t \in \mathbb{R}$.

(a) Notice that C is a curve in the $y = 2$ plane. Compute $\lim_{t \to -\infty} \mathbf{c}(t)$.

Solution: $\lim_{t \to -\infty} \mathbf{c}(t) = (0, 2, 0)$.

(b) Geometrically, C is a curve in the $y = 2$ plane that spirals around the point P that you found in part (a). To see what C looks like, go to `https://www.geogebra.org/3d?lang=en` and type in:

$$\text{Curve}(e^t \cos(t), 2, e^t \sin(t), t, -10, 10)$$

Here, you will see the curve for t-values between -10 and 10. You can change these numbers to see more or less of the curve. You will notice that it seems like the curve begins at P. But if you zoom towards P, you will see how the curve spirals around the y-axis, getting closer and closer to P. There is nothing to hand in for this part.

(c) Show that the reparametrizion of C with respect to arc length starting at $(1, 2, 0)$ in the direction of increasing t is $\mathbf{c}(s) = \left((\frac{s}{\sqrt{2}} + 1) \cos(\ln(\frac{s}{\sqrt{2}} + 1)), 2, (\frac{s}{\sqrt{2}} + 1) \sin(\ln(\frac{s}{\sqrt{2}} + 1)) \right)$.

Solution: At $(1,2,0)$, $t=0$. The arc length function starting at $t=0$ is

$$s(t) = \int_0^t ||\mathbf{c}'(u)||\,du = \int_0^t ||(e^u\cos u - e^u\sin u, 0, e^u\sin u + e^u\cos u)||\,du = \int_0^t \sqrt{2}e^u\,du = \sqrt{2}(e^t-1)$$

Now solving $s=\sqrt{2}(e^t-1)$ for t, we have $t=\ln(\frac{s}{\sqrt{2}}+1)$. Plugging this into the original parametrization, we have

$$\mathbf{c}(s) = \Big((\frac{s}{\sqrt{2}}+1)\cos(\ln(\frac{s}{\sqrt{2}}+1)), 2, (\frac{s}{\sqrt{2}}+1)\sin(\ln(\frac{s}{\sqrt{2}}+1))\Big)$$

(d) Without appealing to the arc length formula, use the reparametrization to find the length of C from $(1,2,0)$ to $\Big((\frac{2}{\sqrt{2}}+1)\cos(\ln(\frac{2}{\sqrt{2}}+1)), 2, (\frac{2}{\sqrt{2}}+1)\sin(\ln(\frac{2}{\sqrt{2}}+1)\Big)$.

Solution: First note that these two points correspond to $s=0$ and $s=2$, respectively. Since the parameter in $\mathbf{c}(s)$ represents the arc length from the starting point $\mathbf{c}(0)$ to $\mathbf{c}(s)$, the arc length from $s=0$ to $s=2$ is simply 2.

4. Let C be the straight line parametrized by $\mathbf{c}(t) = (at+x_0, bt+y_0, ct+z_0)$, where a,b,c,x_0,y_0,z_0 are constants and either $a\neq 0, b\neq 0$ or $c\neq 0$. Show that C has constant curvature equal to 0.

Solution: $\mathbf{c}'(t)=(a,b,c)$ and $\mathbf{c}''(t)=(0,0,0)$. Thus $\mathbf{c}'(t)\times\mathbf{c}''(t)=(0,0,0)$ for all t and so $\kappa(t) = \dfrac{||\mathbf{c}'(0)\times\mathbf{c}''(0)||}{||\mathbf{c}'(0)||^3} = \dfrac{0}{\sqrt{a^2+b^2+c^2}} = 0$ for all t. Thus C has constant curvature equal to 0.

5. In class we saw that the function

$$\mathbf{r}(u,v) = (\sin u, (2+\cos u)\cos v, (2+\cos u)\sin v), \quad 0\le u\le 2\pi, \quad 0\le v\le 2\pi$$

parametrizes a torus T, which is depicted below.

(a) Calculate $||\mathbf{r}_u\times\mathbf{r}_v||$.

Solution:
$\mathbf{r}_u = (\cos u, -\sin u\cos v, -\sin u\sin v)$

$\mathbf{r}_v = (0, -(2+\cos u)\sin v, (2+\cos u)\cos v)$

$\mathbf{r}_u\times\mathbf{r}_v = (-(2+\cos u)\sin u\cos^2 v - (2+\cos u)\sin u\sin^2 v, -(2+\cos u)\cos u\cos v, -(2+\cos u)\cos u\sin v) = (-(2+\cos u)\sin u, (2+\cos u)\cos u\cos v, -(2+\cos u)\cos u\sin v)$

$||\mathbf{r}_u\times\mathbf{r}_v|| = \sqrt{(2+\cos u)^2\sin^2 u + (2+\cos u)^2\cos^2 u\cos^2 v + (2+\cos u)^2\cos^2 u\sin^2 v}$
$= \sqrt{(2+\cos u)^2\sin^2 u + (2+\cos u)^2\cos^2 u} = \sqrt{(2+\cos u)^2} = 2+\cos u.$

(b) Show that T is smooth.

Solution: Since $||\mathbf{r}_u \times \mathbf{r}_v|| = 2 + \cos u \geq 2 + -1 = 1$ for all u we have that $||\mathbf{r}_u \times \mathbf{r}_v|| \neq 0$ for all (u, v). thus $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all u, v and so S is smooth.

(c) Find the equation of the tangent plane to T at $(0, \frac{\pi}{4})$.

Solution: The tangent plane passes through $\mathbf{r}(0, \frac{\pi}{4}) = (0, \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$ and has normal vector $\mathbf{r}_u(0, \frac{\pi}{4}) \times \mathbf{r}_v(0, \frac{\pi}{4})$.

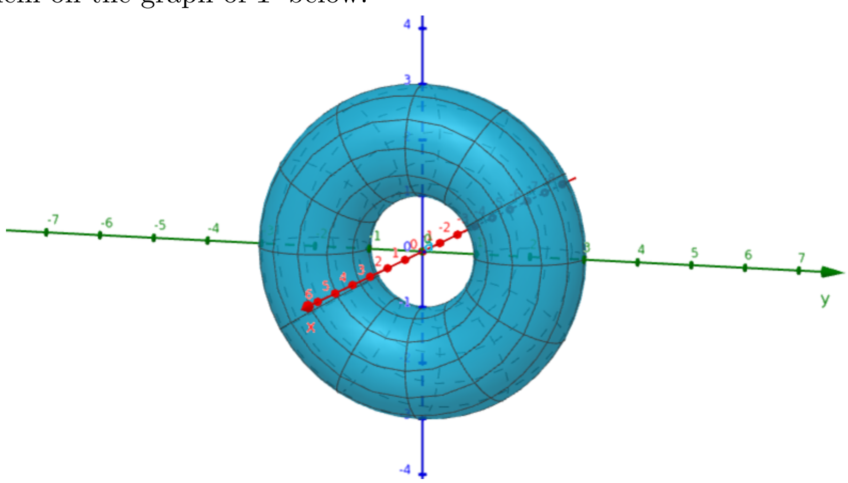
Since $\mathbf{r}_u = (\cos u, -\sin u \cos v, -\sin u \sin v)$ and $\mathbf{r}_v = (0, -(2+\cos u)\sin v, (2+\cos u)\cos v)$, we have $\mathbf{r}_u(0, \frac{\pi}{4}) = (1, 0, 0)$ and $\mathbf{r}_v(0, \frac{\pi}{4}) = (0, -\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$. Thus $\mathbf{r}_u(0, \frac{\pi}{4}) \times \mathbf{r}_v(0, \frac{\pi}{4}) = (0, -\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$.

Thus the equation of the tangent plane is $0(x - 0) - \frac{3\sqrt{2}}{2}(y - \frac{3\sqrt{2}}{2}) - \frac{3\sqrt{2}}{2}(z - \frac{3\sqrt{2}}{2}) = 0$ which simplifies to $3\sqrt{2}y + 3\sqrt{2}z - 36 = 0$

(d) Find the surface area of T.

Solution: $\displaystyle\int_0^{2\pi}\int_0^{2\pi} ||\mathbf{r}_u \times \mathbf{r}_v||\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} (2 + \cos u)\, du\, dv = 8\pi^2.$

(e) Earlier in the semester, we observed that the torus can be built out of one index 0 critical point, two index 1 critical points, and one index 2 critical point. We will now show this concretely. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the height function defined by $h(x, y, z) = z$. When h is restricted to T, it is of form $h(u, v) = (2 + \cos u)\sin v$. Find the critical points of h and classify them using the Hessian. Then evaluate the parametrization $\mathbf{r}$ at the critical points and plot them on the graph of T below.



Solution: $\dfrac{\partial h}{\partial u} = -\sin u \sin v$ and $\dfrac{\partial h}{\partial u} = (2 + \cos u)\cos v$. Now $\dfrac{\partial h}{\partial u} = 0$ if $u = 0, \pi$ or $v = 0, \pi$, while $\dfrac{\partial h}{\partial v} = 0$ if $v = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ (note that $(2 + \cos u) \neq 0$). Thus there are four critical points:

$(0, \frac{\pi}{2})$, $(\pi, \frac{\pi}{2})$, $(0, \frac{3\pi}{2})$ and $(\pi, \frac{3\pi}{2})$.

Now the Hessian of h is $Hh(u, v) = \begin{bmatrix} -\cos u \sin v & -\sin u \cos v \\ -\sin u \cos v & -(2 + \cos u) \sin v \end{bmatrix}$.

At $(0, \frac{\pi}{2})$, $Hh(0, \frac{\pi}{2}) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ has two negative eigenvalues and is thus negative-definite. Thus h has a local max at $(0, \frac{\pi}{2})$ (index 2).

At $(\pi, \frac{\pi}{2})$, $Hh(\pi, \frac{\pi}{2}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has one negative eigenvalue and is thus neither positive- nor negative-definite. Thus h has a saddle point at $(\pi, \frac{\pi}{2})$ (index 1).

At $(0, \frac{3\pi}{2})$, $Hh(0, \frac{3\pi}{2}) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ has two positive eigenvalues and is thus positive-definite. Thus h has a local min at $(0, \frac{3\pi}{2})$ (index 0).

At $(\pi, \frac{3\pi}{2})$, $Hh(0, \frac{\pi}{2}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ has one negative eigenvalue and is thus neither positive- nor negative-definite. Thus h has a saddle point at $(\pi, \frac{3\pi}{2})$ (index 1).

Now, $\mathbf{r}(0, \frac{\pi}{2}) = (0, 0, 3)$, $\mathbf{r}(\pi, \frac{\pi}{2}) = (0, 0, 1)$, $\mathbf{r}(0, \frac{3\pi}{2}) = (0, 0, -3)$, $\mathbf{r}(\pi, \frac{3\pi}{2}) = (0, 0, -3)$. These points can been seen to be the obvious max, min, and saddle points in the graph above.

6. The surfaces parametrized by

$$\mathbf{r}(u, v) = \Big((1+\frac{1}{5}\sin(mu)\sin(nv))\cos u \sin v, (1+\frac{1}{5}\sin(mu)\sin(nv))\sin u \sin v, (1+\frac{1}{5}\sin(mu)\sin(nv))\cos v\Big)$$

where n, m are constants, $0 \leq u \leq 2\pi$, and $0 \leq v \leq \pi$ have been used as models for tumors.

Type the following command into Geogebra's 3D Calculator at `https://www.geogebra.org/3d?lang=en`:

$$\text{Surface}\Big(1 + \frac{1}{5}\sin(4u)\sin(5v))\cos(u)\sin(v), (1 + \frac{1}{5}\sin(4u)\sin(5v))\sin(u)\sin(v),$$
$$(1 + \frac{1}{5}\sin(4u)\sin(5v))\cos(v), u, 0, 2\pi, v, 0, \pi\Big)$$

to see the surface when $m = 4$ and $n = 5$. Be sure to include all parentheses. You can also try to copy and paste it, but it might not work. There is nothing to hand in for this problem. It's just a neat parametrization I wanted to share with you.

7. (NOT TO BE TURNED IN) Here is some additional practice from the textbook:
Section 2.4 # 1, 3, 9, 11,
Section 4.2 # 1,3,17(d),19,
Section 7.3 # 3, 5, 7,

Section 7.4 # 1, 3, 5, 17.
Feel free to do even more problems from the textbook for more practice.