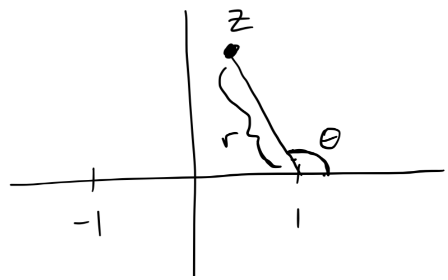
Math 421 HW 4 Additional Problems

1. Consider the multiple-valued function $f(z) = \log(z-1)$. Let $z - 1 = re^{i\theta}$ and define $F(z) = \log(z-1) = \ln r + i\theta$, where $-\pi < \theta \leq \pi$. Let $B = \{z \in \mathbb{C} \mid \text{Re}z \leq -1 \text{ and } \text{Im}z = 0\}$ and $U = \mathbb{C} - B$.

   (a) Draw the complex plane and plot an arbitrary point z is in the first quadrant. Based on your diagram, plot $z-1$ and label r and θ, appropriately. Use your diagram to show that the line segment from z to 1 has length r and the angle formed by the line segment and the positive real axis is θ, as shown below.

   Draw similar diagrams when z is in the second, third, and fourth quadrants.

   (b) Using the diagrams from part (a), explain what happens to θ as z approaches B from above the real axis and below the real axis.

   (c) Conclude that F is not continuous on B.

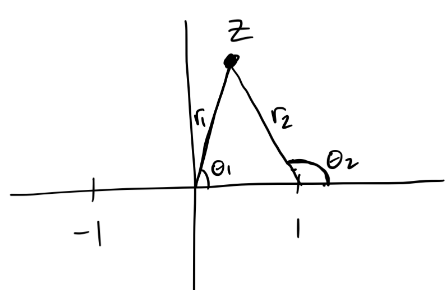
   (d) Use the polar form of the Cauchy-Riemann equations to show that F is differentiable at all points in U. Conclude that F is analytic on U.

   (e) Explain why B is a branch cut of f and $z = 1$ is a branch point of f.

   (f) (Optional) If you are artistically inclined, glue the branches of f together along their branch cuts and sketch the resulting Riemann surface. This is the graph of the imaginary part of f.
   (Visit `https://www.geogebra.org/m/nnjbzvt2` to see the graph of the imaginary part of $\log z$. The graph of f is the same, only shifted one unit along the real axis)

2. Consider the multiple-valued function $g(z) = \log(z) - \log(z-1)$. Let $z = r_1 e^{i\theta_1}$ and $z-1 = r_2 e^{i\theta_2}$ and define $G(z) = \log(z) - \log(z-1) = (\ln r_1 - \ln r_2) + i(\theta_1 - \theta_2)$, where $-\pi < \theta_1, \theta_2 \leq \pi$. Let $B = \{z \in \mathbb{C} \mid 0 \leq \text{Re}z \leq 1 \text{ and } \text{Im}z = 0\}$ and let $U = \mathbb{C} - B$.

   (a) Let z be a point in the first quadrant. Using your answer to problem 1(a), show that $r_1, \theta_1, r_2, \theta_2$ are as in the diagram below.

   Draw similar diagrams when z is in the second, third, and fourth quadrants.

(b) Using the diagrams from part (a), explain what happens to $\theta_1 - \theta_2$ as:

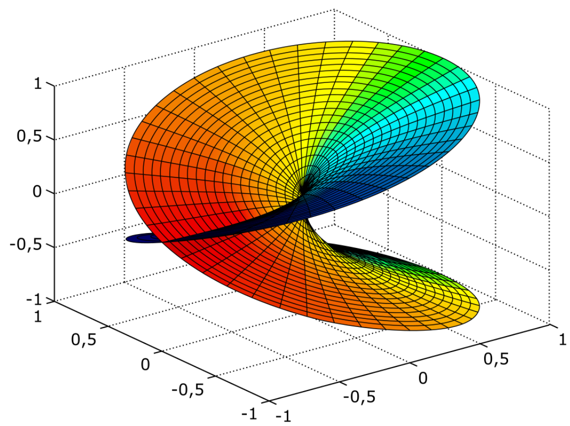
- z approaches B from above the real axis and below the real axis;
- z approaches $B' = \{z \in \mathbb{C} \mid \text{Re}z > 1 \text{ and } \text{Im}z = 0\}$ from above and below;
- z approaches $B'' = \{z \in \mathbb{C} \mid \text{Re}z < 0 \text{ and } \text{Im}z = 0\}$ from above and below;

(c) Conclude that G is not continuous on B.

(d) As in problem 1, it turns out that G is analytic on U. Thus B is a branch cut of g. Moreover, it is easy to see that $z = 1$ and $z = -1$ are branch points of g. There is nothing to do for this part.

(e) (Optional) Once again, if you are artistically inclined, glue the branches of g together along their branch cuts and sketch the resulting Riemann surface.

3. Let $h(z) = z^{\frac{1}{2}}$. Then h is multiple-valued and the negative real axis is a branch cut for h. The graph of the imaginary part of h is the Riemann surface depicted below.



To play around with this graph and with the graph of the real part of h, visit `https://www.geogebra.org/m/FcN24PZ9`. Note that there is nothing to hand in for this question.