Math 425 (Sections 1 and 3) Homework 4 Solutions

1. Determine whether the following vector fields are conservative. Explain your reasoning.

   (a) $\mathbf{F}(x,y,z) = e^x \sin y\mathbf{i} + z\mathbf{j} + 2x\mathbf{k}$

   Solution: Since $\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & z & 2x \end{vmatrix} = (-1, -2, -e^x \cos y) \neq \mathbf{0}$,
   $\mathbf{F}$ is not conservative.

   (b) $\mathbf{F}(x,y) = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j}$

   Solution: Let $P(x,y) = y\cos(xy)$ and $Q(x,y) = x\cos(xy)$. Then $\dfrac{\partial P}{\partial y} = \cos(xy) - xy\sin(xy)$ and $\dfrac{\partial Q}{\partial x} = \cos(xy) - xy\sin(xy)$. Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and $\mathbf{F}$ is a C^1 function defined on all of $\mathbb{R}^2$, which is simply connected, $\mathbf{F}$ is conservative.

   Alternatively, let $f(x,y) = \cos(xy)$. Then $\nabla f = \mathbf{F}$ and so $\mathbf{F}$ is conservative.

2. A vector field is called *incompressible* if its divergence equals 0 at all points and it is called *irrotational* if its curl equals to 0 at all points. Find an example of a nonconstant vector field defined on all of $\mathbb{R}^3$ that is both incompressible and irrotational.

   Solution: Let $\mathbf{F}(x,y,z) = (yz, xz, xy)$. Then $\text{div}\mathbf{F} = 0$ and $\text{curl}\mathbf{F} = \mathbf{0}$.

3. Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$.

   (a) Use the definition of the line integral to compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the unit circle oriented *clockwise*.

   Solution: We can parametrize C by $c(t) = (\cos t, -\sin t)$, $0 \leq t \leq 2\pi$. Thus

   $$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} (\sin t, \cos t) \cdot (-\sin t, -\cos t) dt = \int_0^{2\pi} -1 dt = -2\pi$$

   .

   (b) Using your calculation in part (a), explain why $\mathbf{F}$ is not conservative.

   Solution: If $\mathbf{F}$ were conservative, then by the Fundamental Theorem of Line Integrals (aka the Gradient Theorem), since C is closed, we would have obtained $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$. Since the integral was not 0, as we found in part (b), $\mathbf{F}$ is not conservative.

(c) Show that $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. Why does this not contradict the fact that $\mathbf{F}$ is not conservative?

Solution: $\dfrac{\partial Q}{\partial x} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$ and $\dfrac{\partial P}{\partial y} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$. Thus $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. Since $\mathbf{F}$ is defined only on $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$, which is not simply connected, we cannot conclude that $\mathbf{F}$ is conservative.

4. The *gravitational force field* describes the force of attraction of an object of mass M on an object of mass m. It is the vector field given by

$$\mathbf{F}(x,y,z) = \Big(-\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}x, -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}y, -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}z\Big)$$

where G is a gravitational constant.

(a) Show that $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$ is a potential function for $\mathbf{F}$.

Solution: Since $\nabla f = \mathbf{F}$, f is a potential function for $\mathbf{F}$.

(b) Let C be an arbitrary path from the point $(3,4,12)$ to the point $(0,3,4)$. Compute the work done by the gravitational field in moving an astronaut along C.

Solution: By the fundamental theorem of line integrals

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} = f(0,3,4) - f(3,4,12) = -\frac{mMG}{5} + \frac{mMG}{13} = -\frac{8mMG}{65}$$

(c) Calculate the flux of $\mathbf{F}$ through the unit sphere, with the positive (outward-pointing) orientation. This is known as *Gauss's Law for Gravity.*

Solution: First, parametrize the unit sphere S with $\mathbf{r}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$, $0 \le u \le \pi$ and $0 \le v \le 2\pi$. Then $\mathbf{r}_u \times \mathbf{r}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)$. Note that these normal vectors points outward from S, since, for example, $\mathbf{r}_u \times \mathbf{r}_v(\frac{\pi}{2}, 0) = (1,0,0)$ point outward from the point $\mathbf{r}(\frac{\pi}{2},0) = (1,0,0)$. Thus this parametrization agrees with the positive/outward-pointing orientation of S. Next note that

$$\mathbf{F}(\mathbf{r}(u,v)) = (-mMG \sin u \cos v, -mMG \sin u \sin v, -mMG \cos u)$$

Thus $\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = -mMG \sin u$ and so the flux of $\mathbf{F}$ through S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv = \int_0^{2\pi}\int_0^{\pi} -mMG \sin u\, du\, dv = -4mMG\pi$$

5. Compute the line integral $\displaystyle\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$, where C consists of the line segments from $(1,0,1)$ to $(2,3,1)$ and from $(2,3,1)$ to $(2,5,2)$.

Solution: Since C is built out of two curves, we need to calculate two line integrals. The first line C_1 can be parametrized by $\mathbf{c}_1(t) = (1-t)(1,0,1) + t(2,3,1) = (t+1, 3t, 1)$, where $0 \le t \le 1$. Notice that, with this parametrization, the orientation of C_1 is correct (it goes from $(1,0,1)$ to $(2,3,1)$ as t increases). Similarly, the second curve C_2 can be parametrized by $\mathbf{c}_2(t) = (2, 2t+3, t+1)$, $0 \le t \le 1$. Now, since $\mathbf{c}_1'(t) = (1,3,0)$ and $\mathbf{c}_2'(t) = (0,2,1)$, we have that

$$\begin{aligned}
&\int_{C_1} (x+yz)\,dx + 2x\,dy + xyz\,dz \\
&= \int_0^1 ((t+1) + 3t(1))1\,dt + \int_0^1 2(t+1)(3)\,dt + \int_0^1 (t+1)(3t)(1)(0)\,dt \\
&= \int_0^1 10t + 7\,dt = 12
\end{aligned}$$

$$\begin{aligned}
&\int_{C_2} (x+yz)\,dx + 2x\,dy + xyz\,dz \\
&= \int_0^1 ((2) + (2t+3)(t+1))(0)\,dt + \int_0^1 2(2)(2)\,dt + \int_0^1 (2)(2t+3)(t+1)(1)\,dt \\
&= \int_0^1 4t^2 + 10t + 14\,dt = \frac{61}{3}
\end{aligned}$$

Thus, we have that

$$\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz = \int_{C_1} (x+yz)\,dx + 2x\,dy + xyz\,dz + \int_{C_2} (x+yz)\,dx + 2x\,dy + xyz\,dz = \frac{97}{3}$$

6. (a) Suppose $f(x,y) = e^{\sqrt{y}}$ is the density of the curve C given by $\mathbf{c}(t) = (2, t^2)$, $0 \le t \le 1$. Find the mass of C.

   Solution: $\displaystyle\int_C f\,ds = \int_0^1 f(\mathbf{c}(t))||\mathbf{c}'(t)||\,dt = \int_0^1 2te^t\,dt = 2$

   (b) Suppose $f(x,y,z) = x^2z + y^2z$ is the density of the surface S, which is the part of the plane $x + y - z = -4$ that lies inside of the cylinder $x^2 + y^2 = 4$. Find the mass of S.

   Solution: First, we can parametrize S by $\mathbf{r}(u,v) = (u, v, u+v+4)$ and let $D = \{(x,y) \in \mathbb{R}^3 \mid u^2 + v^2 \le 4\}$. Then $||\mathbf{r}_u \times \mathbf{r}_v|| = \sqrt{3}$. After we setup the integral, we will switch to polar coordinates for easier evaluation.

$$\begin{aligned}
&\iint_S f\,dS = \iint_D f(\mathbf{r}(u,v))||\mathbf{r}_u \times \mathbf{r}_v||\,dA = \iint_D \sqrt{3}(u^2(u+v-4) + v^2(u+v+4))\,dA \\
&= \iint_D \sqrt{3}(u^2+v^2)(u+v+4)\,dA = \int_0^{2\pi}\int_0^2 \sqrt{3}r^2(r\cos\theta + r\sin\theta + 4)r\,dr\,d\theta \\
&= \sqrt{3}\int_0^{2\pi}\int_0^2 r^4\cos\theta + r^4\sin\theta + 4r^3\,dr\,d\theta = 32\pi\sqrt{3}
\end{aligned}$$

7. Let S be the cylinder parametrized by $\mathbf{r}(u,v) = (\cos u, \sin u, v)$, where $0 \le u \le 2\pi$ and $0 \le v \le 1$. Let $\mathbf{F}(x,y,z) = (y, -x, z)$.

(a) Is the orientation on S inward-pointing or outward-pointing? Explain your reasoning.

Solution: Since $\mathbf{r}_u = (-\sin u, \cos u, 0)$ and $\mathbf{r}_v = (0, 0, 1)$, the normal vector to S at $\mathbf{r}(u, v)$ is $\mathbf{r}_u \times \mathbf{r}_u = (\cos u, \sin u, 0)$. When $u = 0$ and $v = 0$, for example, we have the vector $\mathbf{r}_u \times \mathbf{r}_v(0, 0) = (1, 0, 0)$. This vector starts at $\mathbf{r}(0, 0) = (1, 0, 0)$ and points out from the cylinder. Thus the vector field is outward-pointing.

(b) Find the flux of $\mathbf{F}$ through S.

Solution:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dv\, du$$

$$= \int_0^{2\pi} \int_0^1 (\sin u, -\cos u, v) \cdot (\cos u, \sin u, 0)\, dv\, du = \int_0^{2\pi} \int_0^1 0\, dv\, du = 0$$

(c) Show that $\mathbf{F}$ is tangent to S at all points.

Solution: Since $\mathbf{r}_u \times \mathbf{r}_u$ is normal to S and $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$, we we computed in part (b), we have that $\mathbf{F}$ is perpendicular to the normal vectors of S at all points. Thus $\mathbf{F}$ is tangent to S at all points.

(d) Use part (c) to geometrically explain your answer in part (b).

Solution: The flux of a vector field through a surface is the measures the flow of the vector field though the surface. Since $\mathbf{F}$ is tangent to S, the vector field does not flow through S. Thus, the flux is 0.

8. Recall the following parametrization of a torus T:

$$\mathbf{r}(u, v) = (\sin u, (2 + \cos u)\cos v, (2 + \cos u)\sin v), \quad 0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$$

(a) Calculate the Gaussian curvature K of T using the formula from class. There is no need to redo the calculations you did on Homework 3. So for example, you can just write down "$||\mathbf{r}_u \times \mathbf{r}_v|| = 2 + \cos u$, by Homework 3," instead of showing the work all over again.

Solution: From Homework 3, we have that:

$$\mathbf{r}_u = (\cos u, -\sin u \cos v, -\sin u \sin v)$$

$$\mathbf{r}_v = (0, -(2 + \cos u)\sin v, (2 + \cos u)\cos v)$$

$$\mathbf{r}_u \times \mathbf{r}_v = (-(2 + \cos u)\sin u, (2 + \cos u)\cos u \cos v, -(2 + \cos u)\cos u \sin v)$$

$||\mathbf{r}_u \times \mathbf{r}_v|| = 2 + \cos u.$

Thus, we have that:

$\mathbf{N} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||} = (-\sin u, -\cos u \cos v, -\cos u \sin v).$

$\mathbf{r}_{uu} = (-\sin u, -\cos u \cos v, -\cos u \sin v)$

$\mathbf{r}_{uv} = (0, \sin u \sin v, -\sin u \cos v)$

$\mathbf{r}_{vv} = (0, -(2 + \cos u) \cos v, -(2 + \cos u) \sin v)$

Thus $H(u, v) = \begin{bmatrix} \mathbf{N} \cdot \mathbf{r}_{uu} & \mathbf{N} \cdot \mathbf{r}_{uv} \\ \mathbf{N} \cdot \mathbf{r}_{vu} & \mathbf{N} \cdot \mathbf{r}_{vv} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (2 + \cos u) \cos u \end{bmatrix}$

Finally, we have that the Gaussian curvature of T is

$$K = \frac{\det H}{||\mathbf{r}_u \times \mathbf{r}_v||^2} = \frac{(2 + \cos u) \cos u}{(2 + \cos u)^2} = \frac{\cos u}{2 + \cos u}$$

(b) Verify the Gauss-Bonnet Theorem for T. That is, show that $\dfrac{1}{2\pi} \iint\limits_T K \, dS = 2 - 2g$, where g is the genus of T.

Solution: First note that, since T has only one hole, it has genus 1. Thus, $2 - 2g = 2 - 2(1) = 0$. Next, we compute the surface integral of $K(u, v) = \dfrac{\cos u}{2 + \cos u}$ over T.

$$\iint\limits_T K \, dS = \int_0^{2\pi} \int_0^{2\pi} K(u, v) ||\mathbf{r}_u \times \mathbf{r}_v|| \, du \, dv = \int_0^{2\pi} \int_0^{2\pi} \frac{\cos u}{2 + \cos u} (2 + \cos u) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^{2\pi} \cos u \, du \, dv = 0$$

Thus $\dfrac{1}{2\pi} \iint\limits_T K \, dS = 0 = 2 - 2g.$

9. (NOT TO BE TURNED IN) Here is some additional practice from the textbook:
Section 4.4 # 19, 21, 23, 25, 29, 33
Section 7.1 # 11, 15
Section 7.2 # 3, 9, 11, 17
Section 7.5 # 1, 3, 9, 11
Section 7.6 # 1, 11, 13
Feel free to do even more problems from the textbook for more practice.