Math 425 (Sections 1 and 3) Homework 5 Solutions

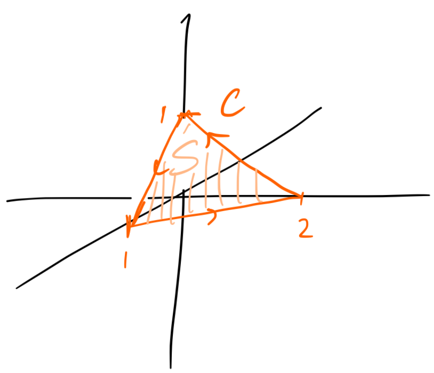
1. Let $\mathbf{F}(x,y,z) = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$. Use Stokes' Theorem to evaluate $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where S is the paraboloid $z = 9 - x^2 - y^2$ that lies above the $z = 5$ plane, oriented upward.

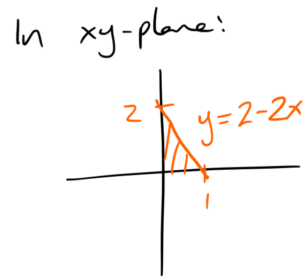
   Solution: The boundary of S is the intersection of the paraboloid and the plane, which is given by $5 = 9 - x^2 - y^2$, or $x^2 + y^2 = 4$. This is a circle of radius 2 in the $z = 5$ plane. Since S has the upward orientation, ∂S has the counterclockwise orientation, when viewed from above. Thus we can paramatrize ∂C by $\mathbf{c}(t) = (2\cos t, 2\sin t, 5)$, $0 \le t \le 2\pi$. Now, $\mathbf{F}(\mathbf{c}(t)) = (10\sin t, -10\cos t, \cos t \sin t)$ and $\mathbf{c}'(t) = (-2\sin t, 2\cos t, 0)$.

   Thus, $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} -20\, dt = -40\pi$

2. Let $\mathbf{F}(x,y,z) = e^{-x}\mathbf{i} + e^x\mathbf{j} + e^z\mathbf{k}$. Use Stokes' Theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant, oriented counterclockwise.

   Solution: Let S be the surface given by the portion of the plane $2x + y + 2z = 2$ in the first octant, with the upward orientation. Then $\partial S = C$, as depicted below.





   Since S is the graph of $z = 1 - x - \frac{1}{2}y$, we can parametrize S by $\mathbf{r}(u,v) = (u, v, 1 - u - \frac{1}{2}v)$, where $0 \le u \le 2$ and $0 \le v \le 2 - 2u$. Since $\mathbf{r}_u \times \mathbf{r}_v = (1, \frac{1}{2}, 1)$ points upward at all points, the induced orientation on C is the correct counterclockwise orientation.

   Now, curl$\mathbf{F}$= $(0, 0, e^x)$ and so curl$\mathbf{F}(\mathbf{r}(u,v)) = (0,0,e^u)$ and curl$\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = e^u$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \text{curl}\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dv\, du = \int_0^1 \int_0^{2-2u} e^u\, dv\, du = 2e - 4$$

3. Let $\mathbf{F}(x,y,z) = x^2y\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$. Use Stokes' Theorem to compute $\int_C \mathbf{F}\cdot d\mathbf{s}$, where C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented clockwise, when viewed from above.

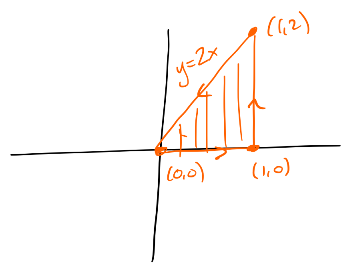
Solution: Let S be the portion of the paraboloid $z = y^2 - x^2$ inside the cylinder $x^2+y^2 = 1$, with the downward orientation. Then $\partial S = C$. Since S is the graph of the function $z = y^2 - x^2$, we can parametrize S by $\mathbf{r}(u,v) = (u, v, v^2 - u^2)$, where $u^2+v^2 \le 1$. However, a quick computation shows that this parametrization does not agree with the downward pointing normal vectors. Thus, we should switch u and v and use the parametrization $\mathbf{r}(u,v) = (v, u, u^2 - v^2)$, where $u^2 + v^2 \le 1$. In this case, we have $\mathbf{r}_u \times \mathbf{r}_v = (-2v, 2u, -1)$, which points downward.

Now, curl$\mathbf{F}$= $(x, -y, 0)$ and so curl$\mathbf{F}(\mathbf{r}(u,v)) = (v, -u, 0)$ and curl$\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = -2v^2 - 2u^2$.

Thus $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D \text{curl}\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv = \int_0^{2\pi}\int_0^1 -2r^3\,dr\,d\theta = -\pi$

4. Let $\mathbf{F}(x,y) = xy\mathbf{i} + x^2y^3\mathbf{j}$. Use Green's Theorem to compute $\int_C \mathbf{F}\cdot d\mathbf{s}$, where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$, oriented counterclockwise.

Solution: The region D bounded by C in the xy-plane, which is depicted below, is given by $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2x\}$

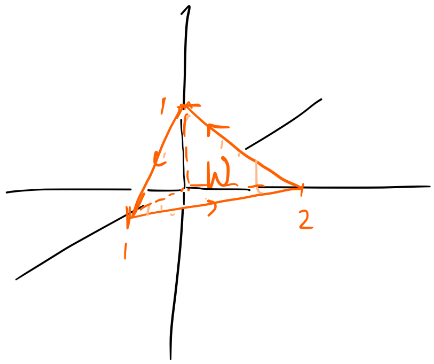


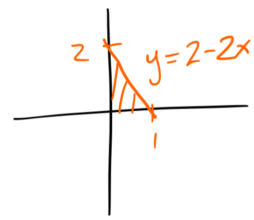
Thus by Green's theorem

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_D \left(\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right)dy\,dx = \int_0^1\int_0^{2x} 2xy^3 - x\,dy\,dx = \frac{2}{3}$$

5. Let $\mathbf{F}(x,y,z) = z^2\mathbf{i} + xy\mathbf{j} + 2xz\mathbf{k}$. Use the Divergence Theorem to compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the surface of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $2x+y+2z = 2$.

Solution: Let W be the tetrahedron with boundary S. Then $W = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x, 0 \le z \le 1 - x - \frac{1}{2}y\}$, which is depicted below.





Since div$\mathbf{F}$= $3x$, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^{2-2x}\int_0^{1-x-\frac{1}{2}y} 3x\,dz\,dy\,dx = \frac{1}{4}$$

6. Let $\mathbf{F} = e^{yz}\mathbf{i} + 3yz^2\mathbf{j} - z^3\mathbf{k}$

   (a) Show that the divergence of $\mathbf{F}$ is 0.

   Solution: div$\mathbf{F}$= $\frac{\partial}{\partial x}(e^{yz}) + \frac{\partial}{\partial y}(3yz^2) + \frac{\partial}{\partial z}(-z^3) = 0 + 3z^2 - 3z^2 = 0$

   (b) Explain why you can deduce that there exists a vector field $\mathbf{G}$ such that $\mathbf{F}$=curl$\mathbf{G}$.

   Solution: Since $\mathbf{F}$ is C^1 on all of $\mathbb{R}^3$, which is simply connected, and div$\mathbf{F} = 0$, there exists a vector field $\mathbf{G}$ such that $\mathbf{F}$=curl$\mathbf{G}$.

   (c) Let S be the unit sphere with the outward-pointing orientation. Use Stokes' Theorem to compute the flux of $\mathbf{F}$ through S.

   Solution: Since S is closed and $\mathbf{F}$ =curl$\mathbf{G}$, for some vector field $\mathbf{G}$, we have that

$$\iint_S \mathbf{F}\,d\mathbf{S} = \iint_S \text{curl}\mathbf{G}\,d\mathbf{S} = 0$$

7. Let D be a simply connected region in $\mathbb{R}^2$ with boundary ∂D, which has the counterclockwise orientation. Let $P(x,y) = 0$ and $Q(x,y) = x$.

   (a) Use Green's theorem to show that the area of a region D is given by $\int_{\partial D} x\,dy$.

   Solution: By Green's theorem, $\int_{\partial D} x\,dy = \int_{\partial D} 0\,dx + x\,dy = \iint_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0)\right) dA =$ $\iint_D 1\,dA$, which is the area of D.

(b) Use the formula from part (a) to find the area of the region bounded by the ellipse $x^2 + 4y^2 = 4$.

Solution: We can parametrize ∂D by $\mathbf{c}(t) = (2\cos t, \sin t)$, $0 \le t \le 2\pi$. Then $\mathbf{c}'(t) = (-2\sin t, \cos t)$. Now, the area is

$$\int_{\partial D} x\,dy = \int_0^{2\pi} 2\cos t(\cos t)\,dt = \int_0^{1} 2\cos^2 t\,dt = \int_0^{1} (1 + \cos(2t))\,dt = 2\pi$$

8. Let $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ and let S_1 and S_2 be the spheres centered at the origin with radii 1 and 2, respectively, oriented outward. Use the Divergence Theorem to show that the flux of $\mathbf{F}$ through S_1 equals the flux of $\mathbf{F}$ through S_2.

Solution: Let W_1 be the unit ball bounded by S_1 and let W_2 be the ball bounded by S_2. Then $\text{div}\mathbf{F} = 0$ and so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iiint_{W_1} \text{div}\mathbf{F}\,dV = \iiint_{W_1} 0\,dV = 0$$

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_{W_2} \text{div}\mathbf{F}\,dV = \iiint_{W_2} 0\,dV = 0$$

Thus the flux of $\mathbf{F}$ through each sphere is 0.

9. (NOT TO BE TURNED IN) Here is some additional practice from the textbook:
Section 8.1 # 7, 9, 15
Section 8.2 # 7, 11, 13, 17
Section 8.4 # 5, 7, 9, 11, 24
Feel free to do even more problems from the textbook for more practice.