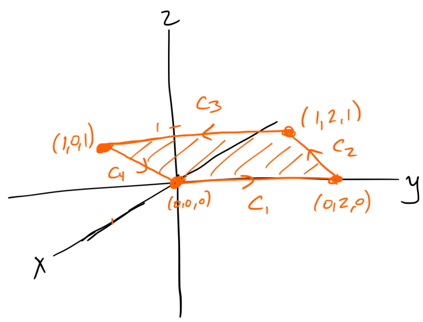
Math 425 (Sections 1 and 3) Homework 6 Solutions

1. Let $\mathbf{F}(x,y,z) = e^{-z^2}\mathbf{i} + 2xy\mathbf{j} + 4y^2\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving a particle along the closed curve formed by the line segments from the origin to $(0,2,0)$, from $(0,2,0)$ to $(1,2,1)$, from $(1,2,1)$ to $(1,0,1)$, and from $(1,0,1)$ to the origin.

   Solution: Computing the work done would be complicated if we use line integrals. Instead, we'll apply Stokes' Theorem. Let C be the closed curve and let C_1, C_2, C_3, and C_4 be the four line segments making up C. Notice that C_1 and C_3 are parallel and C_2 and C_4 are parallel and there plane that contains all four lines. Let S be that plane, as depicted below.

   Notice that C_4, which is the line segment from $(1,0,1)$ to $(0,0,0)$, lies in the xz-plane and is given by $z = x$. Thus the plane S is also given by the equation $z = x$, where $0 \le x \le 1$ and $0 \le y \le 2$. We can thus parametrize S by $r(u,v) = (u,v,u)$, where $0 \le u \le 1$ and $0 \le v \le 2$. Now, $r_u \times r_v = (-1,0,1)$, which points upward from the plane. The induced orientation on C is thus counterclockwise, when viewed from above. However, in the statement of the problem, C is oriented clockwise, when viewed from above. Since changing the orientation negates surface integrals of vector fields, we will simply negate our surface integral when we apply Stokes' Theorem. Note that we could instead switch u and v in the parametrization to reverse the orientation.

   Now, curl$\mathbf{F}$= $(8y, -2ze^{-z^2}, 2y) \Rightarrow \text{curl}\mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (8v, -2ue^{-u^2}, 2v) \cdot (-1,0,1) = -6v$.

   Thus $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} = -\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = -\int_0^1 \int_0^2 -6v\, dv\, du = 12.$

2. Recall, from Homework 4, that the gravitation force field of an object of mass M on an object of mass m is given by

   $$\mathbf{F}(x,y,z) = \Big(-\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}x, -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}y, -\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}z \Big)$$

   where G is a gravitational constant. In Homework 4, you showed that

   $$\iint_S \mathbf{F} \cdot d\mathbf{S} = -4mMG\pi$$

   where S is the sphere circle, oriented outward.

(a) Let W denote the unit ball. Then $\partial W = S$ is the unit sphere. Why can't you apply the Divergence Theorem with W to compute $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$?

Solution: Since $\mathbf{F}$ is not defined at the origin, which is contained in W, we cannot apply the Divergence Theorem.

(b) What would you get if you did try to apply the Divergence Theorem? Is it different from the correct answer?

Solution: If we were to apply the Divergence Theorem without regard to the domain of $\mathbf{F}$, we would have that

$$\mathrm{div}\mathbf{F} = \frac{\partial}{\partial x}\Big(-\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}x\Big)+\frac{\partial}{\partial y}\Big(-\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}y\Big)+\frac{\partial}{\partial z}\Big(-\frac{mMG}{(x^2+y^2+z^2)^{\frac{3}{2}}}z\Big) = 0$$

and so

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_W \mathrm{div}\mathbf{F}\, dV = \iiint\limits_W 0\, dV = 0$$

which is incorrect, since we know $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = -4mMG\pi$.

(c) Let S' be any arbitrary closed surface enclosing S, oriented outward. Show that the flux of $\mathbf{F}$ through S' is also $-4mMG\pi$.

Solution: Let W be the region enclosed by S and S'. Then $\partial W = (-S) \cup (S')$, where $-S$ is the unit sphere oriented inward. Thus, by the Divergence Theorem

$$\iint\limits_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_W \mathrm{div}\mathbf{F}\, dV = \iiint\limits_W 0\, dV = 0$$

On the other hand,

$$\iint\limits_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{-S} \mathbf{F} \cdot d\mathbf{S} + \iint\limits_{S'} \mathbf{F} \cdot d\mathbf{S} = -\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} + \iint\limits_{S'} \mathbf{F} \cdot d\mathbf{S}$$

Thus

$$-\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} + \iint\limits_{S'} \mathbf{F} \cdot d\mathbf{S} = 0$$

or

$$\iint\limits_{S'} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} = -4mMG\pi$$

3. Compute the exterior derivatives of the following differential forms on $\mathbb{R}^3$. Simplify your answers so that they are of the form $f_1(x,y,z)\, dx\, dy + f_2(x,y,z)\, dx\, dz + f_1(x,y,z)\, dy\, dz$.

(a) $\alpha = x\,dx + xyz\,dy$

Solution:

$$\begin{aligned}
d\alpha &= \Big(\frac{\partial}{\partial x}(x)\,dx + \frac{\partial}{\partial y}(x)\,dy + \frac{\partial}{\partial z}(x)\,dz\Big)\,dx + (\frac{\partial}{\partial x}(xyz)\,dx + \frac{\partial}{\partial y}(xyz)\,dy + \frac{\partial}{\partial z}(xyz)\,dz\Big)\,dy \\
&= (\,dx)\,dx + (yz\,dx + xz\,dy + xy\,dz)\,dy \\
&=\ dx\,dx + yz\,dx\,dy + xz\,dy\,dy + xy\,dz\,dy \\
&= -x^2\,dx\,dy + yz\,dx\,dy + xy\,dz\,dy \\
&= yz\,dx\,dy - xy\,dy\,dz
\end{aligned}$$

(b) $\beta =\ dx\,dy + \sin(xz)\,dx\,dz$

Solution:

$$\begin{aligned}
d\beta &= \Big(\frac{\partial}{\partial x}(1)\,dx + \frac{\partial}{\partial y}(1)\,dy + \frac{\partial}{\partial z}(1)\,dz\Big)\,dx\,dy + (\frac{\partial}{\partial x}(\sin(xz))\,dx + \frac{\partial}{\partial y}(\sin(xz))\,dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{\partial}{\partial z}(\sin(xz))\,dz\Big)\,dx\,dz \\
&= (0)\,dx\,dy + (z\cos(xz)\,dx + x\cos(xz)\,dz)\,dx\,dz \\
&= z\cos(xz)\,dx\,dx\,dz + x\cos(xz)\,dz\,dx\,dz \\
&= 0
\end{aligned}$$

4. Let $\alpha = xy\,dx + z\,dy - 3x^2y\,dz$ and $\beta = 2\,dx\,dy - 3e^{xy}\,dx\,dz$ be differential forms on $\mathbb{R}^3$. Calculate $\alpha \wedge \beta$ and simplify your answer so that it is of the form $f_1(x, y, z)\,dx\,dy\,dz$.

Solution:

$$\begin{aligned}
\alpha \wedge \beta &= (xy\,dx + z\,dy - 3x^2y\,dz) \wedge (2\,dx\,dy - 3e^{xy}\,dx\,dz) \\
&= 2xy\,dx\,dx\,dy + 2z\,dy\,dx\,dy - 6x^2y\,dz\,dx\,dy - 3xye^{xy}\,dx\,dx\,dz - 3ze^{xy}\,dy\,dx\,dz + 9x^2ye^{xy}\,dz\,dx\,dz \\
&= -6x^2y\,dz\,dx\,dy - 3ze^{xy}\,dy\,dx\,dz \\
&= (3ze^{xy} - 6x^2y)\,dx\,dy\,dz
\end{aligned}$$

5. A 1-form α on $\mathbb{R}^3$ is called a *contact 1-form* if $\alpha \wedge d\alpha \neq 0$ at all points in $\mathbb{R}^3$. Show that $\alpha = dz - xdy$ is a contact 1-form on $\mathbb{R}^3$.

Solution: Since $d\alpha = \Big(\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(1)\Big)\,dz - \Big(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(x)\Big)\,dy = -\,dx\,dy$, we have that $\alpha \wedge\ d\alpha = (\,dz - x\,dy) \wedge (-\,dx\,dy) = -\,dz\,dx\,dy + x\,dy\,dx\,dy = -\,dz\,dx\,dy \neq 0$. Thus α is a contact structure.

6. Determine whether the 1-form $\alpha = y^3z^2\,dx + 3xy^2z^2\,dy + 2xy^3z\,dz$ on $\mathbb{R}^3$ is exact. Explain your reasoning. If it is exact, express it as df, where f is some 0-form.

Solution: Let $\mathbf{F} = (y^3z^2, 3xy^2z^2, 2xy^3z)$. Then since $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ (which is simply connected), β is exact if and only if curl$\mathbf{F}$= $\mathbf{0}$. Now

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix} = (0,0,0)$$

Thus β is exact. By integrating y^3z^2, $3xy^2z^2$, and $2xy^3z$ with respect to x, y, and z, respectively, it is easy to see that if $f(x,y,z) = xy^3z^2$, then $df = \beta$.

Alternatively, since α is closed (check) and α is defined on $\mathbb{R}^3$, by the Poincaré Lemma, α is exact.

7. Show that every k-form on $\mathbb{R}^k$ is closed. (Hint: Notice that a k-form on $\mathbb{R}^k$ is necessarily of the form $\nu = f(x_1, \ldots, x_k)\, dx_1 \cdots dx_k$.)

Solution:

$$d\nu = \sum_{i=1}^{k} \Big(\frac{\partial f}{\partial x_i}\, dx_i \Big)\, dx_1 \cdots dx_k = \sum_{i=1}^{k} (-1)^{i-1} \frac{\partial f}{\partial x_i}\, dx_1 \cdots \underbrace{dx_i\, dx_i}_{=0} \cdots dx_k = \sum_{i=1}^{k} 0 = 0$$

8. Let $\alpha = -\dfrac{y}{x^2+y^2}\, dx + \dfrac{x}{x^2+y^2}\, dy$ be a 1-form on $\mathbb{R}^2$.

(a) Show that α is closed.

Solution: α is closed since

$$\begin{aligned} d\alpha &= \Big(\frac{\partial}{\partial x}(-\frac{y}{x^2+y^2})\, dx + \frac{\partial}{\partial y}(-\frac{y}{x^2+y^2})\, dy \Big)\, dx + \Big(\frac{\partial}{\partial x}(\frac{x}{x^2+y^2})\, dx + \frac{\partial}{\partial y}(\frac{x}{x^2+y^2})\, dy \Big)\, dy \\ &= \frac{\partial}{\partial y}(-\frac{y}{x^2+y^2})\, dy\, dx + \frac{\partial}{\partial x}(\frac{x}{x^2+y^2})\, dx\, dy \\ &= \frac{y^2-x^2}{(x^2+y^2)^2}\, dy\, dx + \frac{y^2-x^2}{(x^2+y^2)^2}\, dx\, dy \\ &= -\frac{y^2-x^2}{(x^2+y^2)^2}\, dx\, dy + \frac{y^2-x^2}{(x^2+y^2)^2}\, dx\, dy = 0 \end{aligned}$$

(b) Show that α is not exact. (Hint: You already showed this on Homework 4. Explain why)

Solution: Let $\mathbf{F}(x,y) = \Big(-\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \Big)$. On Homework 3, we showed that $\mathbf{F}$ is not conservative, which is equivalent to α being not exact.

(c) Why doesn't this contradict the Poincaré Lemma?

Solution: The Poincaré Lemma states that if a differential form ω is closed and defined on all of $\mathbb{R}^n$, then ω is closed. In this case, α is closed, but not defined at $(0,0)$. Thus the lemma is not applicable.

9. Let ω be a closed $(k-1)$-form defined on a k-dimensional manifold M with boundary ∂M. Use the generalized Stokes' Theorem to compute $\displaystyle\int_{\partial M} \omega$.

   Solution: Since ω is closed, $d\omega = 0$. Thus, by Stokes' Theorem,

$$\int\limits_{\partial M} \omega = \int\limits_{M} d\omega = \int\limits_{M} 0 = 0$$

10. Let $\omega = P(x,y)\,dx + Q(x,y)\,dy$ be a 1-form on $\mathbb{R}^2$.
    (a) Calculate and simplify $d\omega$.

    Solution:

$$\begin{aligned} d\omega &= \Big(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy\Big)\,dx + \Big(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy\Big)\,dy \\ &= \frac{\partial P}{\partial x}\,dx\,dx + \frac{\partial P}{\partial y}\,dy\,dx + \frac{\partial Q}{\partial x}\,dx\,dy + \frac{\partial Q}{\partial y}\,dy\,dy \\ &= \frac{\partial P}{\partial y}\,dy\,dx + \frac{\partial Q}{\partial x}\,dx\,dy \\ &= \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy \end{aligned}$$

    (b) Let D be a region in $\mathbb{R}^2$ with oriented boundary ∂D. Use the Generalized Stokes' Theorem to prove Green's Theorem:

$$\iint\limits_{D} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy = \int\limits_{\partial D} P\,dx + Q\,dy$$

    Solution:

$$\iint\limits_{D} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy = \int\limits_{D} d\omega = \int\limits_{\partial D} \omega = \int\limits_{\partial D} P\,dx + Q\,dy$$

11. (NOT TO BE TURNED IN) Here is some additional practice from the textbook:
    Section 8.5 # 1, 3, 5, 6, 11, 13
    Feel free to do even more problems from the textbook for more practice.