Math 421 Homework 8

1. Suppose f is entire and $|f(z)| \geq 4$ for all $z \in \mathbb{C}$. Prove that f is constant on $\mathbb{C}$.

2. Let $f(z) = z + e^z$.
   (a) What does the Maximum Modulus Principle tell you about f?
   (b) Express f in the form $f(z) = u(x, y) + iv(x, y)$.
   (c) In class we showed that, as a consequence of the Maximum Modulus Principle, the only critical points of u are saddle points. Let's show this concretely with the example in this problem.
      1. Write down the statement of the second derivative test from Calc III (Math 233).
      2. Use the second derivative test to show that the only critical points of u are saddle points.

3. Let $f(z) = u(x, y) + iv(x, y)$ and suppose $f'(z_0)$ exists. Recall that z_0 is a critical point of f if $f'(z_0) = 0$. We will show that $z_0 = x_0 + iy_0$ is a critical point of f if and only if (x_0, y_0) is a critical point of u if and only if (x_0, y_0) is a critical point of v.
   (a) Show that if (x_0, y_0) is a critical point of u if and only if (x_0, y_0) is a critical point of v.
   (b) Show that if (x_0, y_0) is a critical point of u, then z_0 is a critical point of f.
   (c) Show that if z_0 is a critical point of f, then (x_0, y_0) is a critical point of both u and v.

4. Use contour integrals to evaluate the following integrals.
   (a) $\displaystyle\int_0^{2\pi} \frac{1}{5 + 4\sin t}\, dt$
   (b) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2}\, dx$