Math 421 Homework 9

1. Consider the sequence $z_n = \left(\frac{2\sqrt{2}}{3-3i}\right)^n$, $n \geq 1$.
   (a) Sketch the sequence in the complex plane. (Hint: rewrite the terms in the sequence in exponential form).
   (b) Does z_n converge or diverge? If it converges, determine what it converges to. Explain you reasoning.

2. Let z_n be a complex sequence. Show that if $\lim_{n\to\infty} z_n = 0$, then $\lim_{n\to\infty} |z_n| = 0$.
   (Hint: Let $z_n = x_n + iy_n$ and use the fact that if $\lim_{n\to\infty} z_n$ exists, then $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$)

3. Use the partial sum definition of series to show that if $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$.

4. Consider the geometric series $\sum_{n=1}^{\infty} z^n$.
   (a) For what values of z is the series convergent and divergent? When convergent, what is the sum of the series? (Note that the series starts at $n = 1$, not $n = 0$)
   (b) Let $z = re^{i\theta}$. Break the series into its real and imaginary parts.
   (c) Use parts (a) and (b) to show that when $0 < r < 1$,

$$\sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \text{ and } \sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

5. Determine whether the following series converge or diverge. Cite any tests that you use.
   (a) $\sum_{n=1}^{\infty} \left(\frac{3}{1-i}\right)^n$
   (b) $\sum_{n=1}^{\infty} \frac{e^{\frac{\pi}{5}ni}}{n^2}$
   (c) $\sum_{n=1}^{\infty} \frac{e^{\frac{\pi}{2}ni}}{n}$
   (d) $\sum_{n=1}^{\infty} \frac{n^2 - in + 1}{2 - (3+i)n^2}$

6. Consider the power series $\sum_{n=0}^{\infty} \frac{i}{2^n}(z+2i)^n$.
   (a) Find the radius and disk of convergence (Hint: use the root or ratio test). Sketch the disk in the complex plane.

(b) Use the divergence test to show that the power series diverges at all points on the boundary of the disk of convergence. (Hint: use problem 2, above)

7. Consider the power series $\sum_{n=0}^{\infty} \frac{1}{(n+1)3^n} z^n$.

   (a) Find the radius and disk of convergence. Sketch the disk in the complex plane.

   (b) Use the Dirichlet test to show that the power series converges at all points on the boundary of the disk of convergence except at $z = 3$. Explain why the series diverges at $z = 3$.

8. (OPTIONAL) Consider the power series $\sum_{n=1}^{\infty} \frac{i}{n^2} z^n$.

   (a) Find the radius and disk of convergence. Sketch the disk in the complex plane.

   (b) Show that the power series converges at all points on the boundary of the disk of convergence by showing it is absolutely convergent.