Pre-Class Problems

To get the most out of class, you will be assigned problems from the list below to prepare you for the upcoming class material. These assignments are to be completed before coming to class. These assignments will either: (1) review pre-requisite concepts from calculus and linear algebra or (2) review material from a previous lecture. This bank of problems will be updated regularly and solutions will be added after each class.

Visit the class website https://people.umass.edu/jsimone/fall2019math425.html for more details, including pre-class assignment due dates.

1. (Calc III Review) Let $f(x, y, z) = x^2 e^{2z-y}$. Compute $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, and $\dfrac{\partial^2 f}{\partial z^2}(1, 2, 1)$.

   Solution: $\dfrac{\partial f}{\partial y} = -x^2 e^{2z-y}$, $\dfrac{\partial^2 f}{\partial x \partial y} = -2xe^{2z-y}$, $\dfrac{\partial^2 f}{\partial z^2}(1, 2, 1) = 4$

2. Find the linearization L of:

   (a) (Calc I Review) $f(x) = \sin x + 1$ at $x = \pi$.

   Solution: $L(x) = f(\pi) + f'(\pi)(x - \pi) = 1 - 1(x - \pi) = 1 + \pi - x$

   (b) (Calc III Review) $g(x, y) = x^2 + y^2 + 1$ at $(x, y) = (1, 1)$

   Solution: $L(x, y) = g(1, 1) + \frac{\partial g}{\partial x}(1, 1)(x - 1) + \frac{\partial g}{\partial y}(1, 1)(y - 1) = 2x + 2y - 1$

3. (Calc III Review) Compute the following limits, if they exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x + 1}$

   Solution: Since $f(x, y) = \frac{e^{xy}}{x+1}$ is continuous at $(0, 0)$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x + 1} = 1$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{x^2 + y^2}$

   Solution: Along the line $x = 0$, when $y \neq 0$, $\frac{(x-y)^2}{x^2+y^2} = \frac{y^2}{y^2} = 1$ and so as $(x, y) \to (0, 0)$ along $x = 0$, $\frac{(x-y)^2}{x^2+y^2} \to 1$. On the other hand, along the line $x = y$, when $(x, y) \neq (0, 0)$, $\frac{(x-y)^2}{x^2+y^2} = 0$ and so as $(x, y) \to (0, 0)$ along $x = y$, $\frac{(x-y)^2}{x^2+y^2} \to 0$. Thus the limit does not exist.

4. (Calc III Review) Let $f(x, y) = 3x^2 + 2\ln(xy)$, $x = 3s + 2t$, $y = se^t$. Use the chain rule to compute $\dfrac{\partial f}{\partial s}$.

   Solution: $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} = (6x + \dfrac{2}{x})(3) + (\dfrac{2}{y})(e^t) = 18(3s + 2t) + \dfrac{6}{3s + 2t} + \dfrac{2}{s}$

5. (Calc III Review) Find and classify the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$.

Solution: $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$. Setting these both to zero and solving, we find the critical points $(0,0)$ and $(1,1)$. Now we can classify them by using the second derivative test from Calc III. First note that $f_{xx} = 6x$, $f_{yy} = 6y$, and $f_{xy} = -3$. Since $f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = -9 < 0$, f has a saddle point at $(0,0)$ and since $f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = 27 > 0$ and $f_{xx}(1,1) = 6 > 0$, f has a local minimum at $(1,1)$.

6. (Linear Algebra Review)

   (a) Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

   Solution: Since A is triangular, the eigenvalues are 1 and -2.

   (b) Fill in the blank: An $n \times n$ matrix M has an eigenvalue of 0 if and only if $\det M =$ ____.

   Solution: 0

   (c) True/False : A symmetric $n \times n$ matrix M (i.e. $M^T = M$) has n real eigenvalues.

   Solution: True

7. (Review of Previous Lecture)

   (a) Write down the general second derivative test from last lecture.

   Solution: Let f be a C^2 function with nondegenerate critical point $\mathbf{x}_0$. If $Hf(\mathbf{x}_0)$ is positive-definite, then f has a local minimum at $\mathbf{x}_0$; if $Hf(\mathbf{x}_0)$ is negative-definite, then f has a local maximum at $\mathbf{x}_0$; otherwise, f has a saddle point at $\mathbf{x}_0$.

   (b) Show that $(0,0,0)$ is a critical point of $f(x,y,z) = x^2 + y^2 + z^2 + yz$. Use the second derivative test to show that it is a local minimum.

   Solution: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y + z$, and $\frac{\partial f}{\partial z} = 2z + y$. Since $\frac{\partial f}{\partial x}(0,0,0) = \frac{\partial f}{\partial y}(0,0,0) = \frac{\partial f}{\partial z}(0,0,0) = 0$, $(0,0,0)$ is a critical point. Computing the second partial derivatives, we have that the Hessian at $(0,0,0)$ is $Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Since $\det[2] = 2 > 0$, $\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0$, and $\det Hf(0,0,0) = 6 > 0$, the Hessian is positive-definite. Thus $(0,0,0)$ is a local minimum.

   (c) Show that $(0,0)$ is a degenerate critical point of $f(x,y) = x^7 - y^4$.

Solution: First, since $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y} = 0$, $(0,0)$ is a critical point. The Hessian is $Hf(x,y) = \begin{bmatrix} 42x^5 & 0 \\ 0 & -12y^2 \end{bmatrix}$. Thus $\det Hf(0,0) = 0$ and so $(0,0)$ is degenerate.

8. (Calculus I Review) State the Extreme Value Theorem for a function of one variable. Find the absolute maximum and minimum values of $f(x) = \sin(2x)$ on $[0, \frac{\pi}{2}]$.

   Solution: The EVT states: If f is continous on $[a,b]$, then f attains an absolute maximum and absolute minimum on $[a,b]$. To find the extrema, we simply find the critical points of f in $[0, \frac{\pi}{2}]$, plug them and the endpoints into f, and compare the function values. Since $f'(x) = 2\cos(2x)$, the only critical point in $[0, \frac{\pi}{2}]$ is $x = \frac{\pi}{4}$. Now, $f(0) = 0, f(\frac{\pi}{4}) = 1$, and $f(\frac{\pi}{2}) = 0$. Thus the absolute maximum is 1 and the absolute minimum is 0.

9. (Calculus III Review) Use Lagrange multipliers to find the extreme values of $f(x,y) = xy$ subject to the constraint $x^2 + y^2 = 1$.

   Solution: Let $g(x,y) = x^2 + y^2$. Then $\nabla f(x,y) = \langle y, x\rangle$ and $\nabla g(x,y) = \langle 2x, 2y\rangle$. We must solve the system: $y = \lambda 2x, x = \lambda 2y, x^2 + y^2 = 1$. First note that if $\lambda = 0$, then $x = y = 0$, which contradicts the third equation. Similary, if $x = 0$, then since $\lambda \neq 0$, the first equation would force $y = 0$, once again contradicting the third equation. Thus $\lambda, x, y \neq 0$. By solving the first two equations for lambda, setting them equal to each other, and cross multiplying, we obtain $x^2 = y^2$. Plugging this into the third equation yields $2x^2 = 1$, or $x = \pm\frac{1}{\sqrt{2}}$. Thus the critical points are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Plugging these into f, we see that the absolute maximum is $\frac{1}{2}$ and the absolute minimum is $-\frac{1}{2}$

10. (Calculus III Review) Evaluate the following double integrals.

    (a) $\displaystyle\int_0^1 \int_{-1}^1 x^2 y\, dx\, dy$

    Solution: $\displaystyle\int_0^1 \int_{-1}^1 x^2 y\, dx\, dy = \int_0^1 \left(\frac{1}{3}x^3 y\Big|_{-1}^1\right) dy = \int_0^1 \frac{2}{3} y\, dy = \frac{1}{3}$

    (b) $\displaystyle\int_0^1 \int_1^{e^x} (x+y)\, dy\, dx$

    Solution: $\displaystyle\int_0^1 \int_1^{e^x} (x+y)\, dy\, dx = \int_0^1 \left(xy + \frac{1}{2}y^2\Big|_1^{e^x}\right) dx = \int_0^1 xe^x + \frac{1}{2}e^{2x} - x - \frac{1}{2}\, dx$

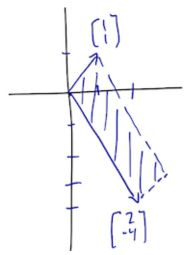
    $\displaystyle = xe^x - e^x + \frac{1}{4}e^{2x} - \frac{1}{2}x^2 - \frac{1}{2}x\Big|_0^1 = \frac{1}{4}e^2 - \frac{1}{4}$

11. (Review of Previous Lecture) Compute $\displaystyle\int_0^1 \int_0^2 \int_0^3 xyz^2\, dz\, dy\, dx$.

    Solution: $\displaystyle\int_0^1 \int_0^2 \int_0^3 xyz^2\, dz\, dy\, dx = \int_0^1 \int_0^2 9xy\, dy\, dx = \int_0^1 18x\, dx = 9$

12. (Linear Algebra Review): Let $A = \begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}$

(a) Sketch the parallelogram formed by the columns of A.



Solution:

(b) Without using geometry, find the area of that parallelogram. (Hint: It is related to the determinant of A.)

Solution: The area is $|\det A| = 6$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A and let $D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$. What is the area of $T(D)$ (i.e. the image of D)? (Hint: It is related to the determinant of A and the area of D).

Solution: $\text{Area}(T(D)) = |\det A|\text{Area}(D) = 6(4) = 24$.

13. (Linear Algebra Review) Recall that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if for each $\mathbf{b} \in \mathbb{R}^m$, there is at most one $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$. Let T be a linear transformation given by a matrix A. If T is one-to-one, then what can be said about the columns of A? What can be said about the determinant of A?

Solution: The columns of A are linearly independent and $\det A \neq 0$.

14. (Review of Previous Lecture) Write down the change of variables formulas for double integrals. Use the change of variables $u = 2x + y$, $v = x - 3y$ to rewrite the integral $\iint (2x + y)e^{x-3y}\, dx\, dy$ in terms of u and v. Don't forget about the determinant of the Jacobian.

Solution: To figure out the Jacobian, we need to write x and y in terms of u and v.
$u = 2x + y \Rightarrow y = u - 2x \Rightarrow v = x - 3y = x - 3u + 6x \Rightarrow x = \frac{3}{7}u + \frac{1}{7}v \Rightarrow y = u - 2x = \frac{1}{7}u - \frac{2}{7}v$.
Thus the change of variables transformation is $T(u, v) = (\frac{3}{7}u + \frac{1}{7}v, \frac{1}{7}u - \frac{2}{7}v)$ and so we have

$$|\det \mathbf{D}T(u, v)| = \left|\det \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix}\right| = \frac{1}{7}$$

Thus we have $\displaystyle\iint (2x + y)e^{x-3y}\, dx\, dy = \iint \frac{1}{7}ue^v\, du\, dv$.

15. (Calculus III Review)

(a) Find the vector equation of the line in $\mathbb{R}^3$ passing through the point $(1, 2, 3)$ and parallel to the vector $\langle -1, 3, 0\rangle$.

Solution: $\mathbf{c}(t) = \langle 1, 2, 3\rangle + t\langle -1, 3, 0\rangle = \langle 1 - t, 2 + 3t, 3\rangle$

(b) Sketch the curve in $\mathbb{R}^2$ given by $\mathbf{c}(t) = (\cos t, \sin t)$, where $0 \le t \le 2\pi$.

Solution: Since $x = \cos t$ and $y = \sin t$, we have that $x^2 + y^2 = 1$ and so the curve is simply the unit circle.

16. (Calculus III Review) Recall the arc length formula: The length of a curve parametrized by $\mathbf{c}(t)$, where $a \le t \le b$ is

$$L = \int_a^b ||\mathbf{c}'(t)||dt$$

Find the length of the curve parametrized by $\mathbf{c}(t) = (r\cos t, r\sin t)$, where $0 \le t \le 2\pi$.

Solution: Since $\mathbf{c}'(t) = (-r\sin t, r\cos t)$, we have that $||\mathbf{c}'(t)|| = r$. Thus $L = \displaystyle\int_0^{2\pi} r\,dt = 2\pi r$.

17. (Calculus III Review) Compute the cross product $\mathbf{a}\times\mathbf{b}$, where $\mathbf{a} = \langle 2, -1, 3\rangle$ and $\mathbf{b} = \langle -2, 3, 5\rangle$. How is it related to the original two vectors?

Solution:
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 3 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix}\mathbf{k} = -14\mathbf{i} - 16\mathbf{j} + 4\mathbf{k} = \langle -14, -16, 4\rangle.$$

$\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

18. (Calculus III Review) Find the equation of the plane passing through $(1, 2, 3)$ with normal vector $\langle -1, 4, 2\rangle$.

Solution: $-1(x - 1) + 4(y - 2) + 2(z - 3) = 0$ or $-x + 4y + 2z - 13 = 0$.

19. (Calculus III Review) Find a vector normal to the plane passing through the points $A(1, 0, 1)$, $B(2, -1, 1)$, and $C(0, 0, 2)$.

Solution: First, we find two vectors parallel to the plane: $\overrightarrow{AB} = \langle 1, -1, 0\rangle$ and $\overrightarrow{AC} = \langle -1, 0, 1\rangle$. Now, to find a normal vector, we can simply compute the cross product of these two vectors: $\overrightarrow{AB}\times\overrightarrow{AC} = \langle -1, -1, -1\rangle$. Note that any multiple of this vector is a normal vector to the plane.

20. (Calculus III Review) Use the cross product to find the area of the parallelogram spanned by the vectors $\mathbf{a} = \langle 2, 1, 3 \rangle$ and $\mathbf{b} = \langle -1, 0, 3 \rangle$.

Solution: The area of parallelogram is $||\mathbf{a} \times \mathbf{b}|| = ||(3, -9, 1)|| = \sqrt{91}$.

21. Try to wrap an orange in a sheet of paper. What happens to the sheet of paper?

Solution: It gets crumpled up.

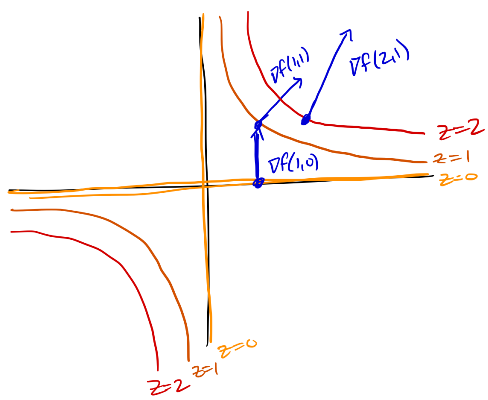
22. (Calculus III Review) Let $f(x, y) = xy$.

(a) Compute the gradient of f.

Solution: $\nabla f(x, y) = \langle y, x \rangle$.

(b) Sketch the level curves of f for $z = 0, 1, 2$ and draw the vectors $\nabla f(1, 0)$, $\nabla f(1, 1)$, $\nabla f(2, 1)$.

Solution: The level curves are given by $0 = xy \Rightarrow x = 0$ or $y = 0$, $1 = xy \Rightarrow y = \frac{1}{x}$, and $2 = xy \Rightarrow y = \frac{2}{x}$. Thegradients are $\nabla f(1, 0) = (0, 1)$, $\nabla f(1, 1) = (1, 1)$, $\nabla f(2, 1) = (1, 2)$.



(c) In general, how are the level sets of a function related to the gradient vectors, geometrically?

Solution: The gradient vectors are perpendicular to the level sets.

23. (Review of Last Class) Let $\mathbf{F}(x, y, z) = (xyz, x^2 \cos y, z^2)$.

(a) Calculate div$\mathbf{F}$.

Solution: div$\mathbf{F}$= $\frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(x^2 \cos y) + \frac{\partial}{\partial z}(z^2) = yz - x^2 \sin y + 2z$

(b) Calculate curl$\mathbf{F}$.

Solution: curl$\mathbf{F}$= $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & x^2\cos y & z^2 \end{vmatrix} = \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ x^2\cos y & z^2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xyz & z^2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xyz & x^2\cos y \end{vmatrix}$

$= (\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x^2\cos y))\mathbf{i} - (\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(xyz))\mathbf{j} + (\frac{\partial}{\partial x}(x^2\cos y) - \frac{\partial}{\partial y}(xyz))\mathbf{k}$

$= (0, xy, 2x\cos y - xz)$

24. (Review of Last Class): Recall that if f is a C^2 function, then curl$(\nabla f) = \mathbf{0}$. Use this fact to show that $\mathbf{F}(x, y, z) = (-y, x, 0)$ is not conservative.

    Solution: Since curl$\mathbf{F} = (0, 0, 2) \neq \mathbf{0}$, $\mathbf{F}$ is not conservative.

25. (Parametrization Review) Let C be the portion of the unit circle in the first quadrant. It can be parametrized by $\mathbf{c}(t) = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}$.

    (a) What is the orientation/direction of this parametrization (i.e., as t increases, do you move clockwise or counterclockwise)?

    Solution: The parametrization moves counterclockwise, since $\mathbf{c}(0) = (1, 0)$ and $\mathbf{c}(\frac{\pi}{2}) = (0, 1)$.

    (b) Show that $\mathbf{c}(t) = (\cos 2t, \sin 2t), 0 \leq t \leq \frac{\pi}{4}$ is also a parametrization of C with the same orientation.

    Solution: Since $x = \cos 2t$ and $y = \sin 2t$, we have $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$. Moreover, $\mathbf{c}(0) = (1, 0)$ and $\mathbf{c}(\frac{\pi}{4}) = (0, 1)$. Thus this parametrization is of the same curve, with the same endpoints, and has the same orientation.

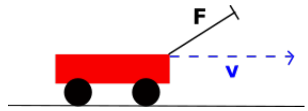
    (c) Reparametrize C so that the orientation is reversed.

    Solution: There are different ways to reverse the orientation. Here are a couple of ways:
    $\mathbf{c}(t) = (\cos(\frac{\pi}{2} - t), \sin(\frac{\pi}{2} - t)), 0 \leq t \leq \frac{\pi}{2}$ or
    $\mathbf{c}(t) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t), \frac{3\pi}{2} \leq t \leq 2\pi$

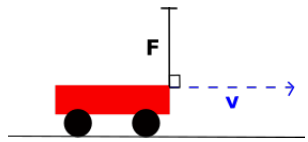
26. (Some Light Physics/Calculus III Review) If $\mathbf{F}$ is a force field in $\mathbf{R}^3$, then an object in $\mathbb{R}^3$ might be moved by $\mathbf{F}$ (e.g. an apple falling from a tree due to a gravitational field). Suppose we want the force field to move an object at rest along a particular path. The **work** done by $\mathbf{F}$ measures how easy it is for $\mathbf{F}$ to move the object along the path.

    If $\mathbf{F}$ is a constant force and the path is a straight line segment given by a vector $\mathbf{v}$, then the work done is given by the dot product $W = \mathbf{F} \cdot \mathbf{v}$.

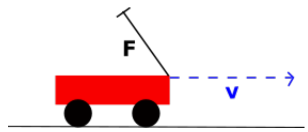
    A classic example is when you pull a wagon. Suppose this wagon is connected to tracks and can only move left or right. The vector $\mathbf{F}$ corresponds to the handle; $\mathbf{F}$ points in the direction in which you are pulling the wagon and $||\mathbf{F}||$ measures how hard you are pulling the wagon. The vector $\mathbf{v}$ is the direction that you want to pull the wagon.



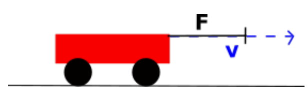
If the work done is 0, then the force is unable to move the object at all. This occurs when $W = \mathbf{F} \cdot \mathbf{v} = 0$, which geometrically means $\mathbf{F}$ is perpendicular to $\mathbf{v}$. In the wagon example, if you pull the handle straight up, it obviously will not move forward.



If $W < 0$, then the force moves the object in the "wrong" direction. This occurs when the angle between $\mathbf{F}$ and $\mathbf{v}$ is greater than $90°$.



If $W > 0$, then the force is able to move the object in the correct direction. The work is maximized when $\mathbf{F}$ and $\mathbf{v}$ point in the same direction. In the wagon example, if you pull the wagon for a minute, you will get the furthest if $\mathbf{F}$ and $\mathbf{v}$ are parallel to each other.



Now for an actual computation: Suppose the handle of the wagon is given by $\mathbf{F} = 2\mathbf{i} + \mathbf{j} = \langle 2, 1 \rangle$ and the direction in which we wish to pull it is $\mathbf{v} = \mathbf{i} = \langle 1, 0 \rangle$. Compute the work done by $\mathbf{F}$ to move the wagon along $\mathbf{v}$.

Solution: $W = \mathbf{F} \cdot \mathbf{v} = 2$.

27. (Review of last lecture) Compute the work done by the force $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in moving a particle along the curve parametrized by $\mathbf{c}(t) = (t^2, 3t, 2t^3)$, $0 \le t \le 1$.

Solution:

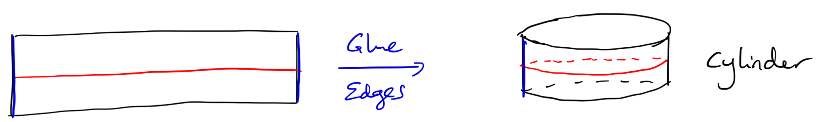
$$W = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 (t^2, 3t, 2t^3) \cdot (2t, 3, 6t^2)\, dt = \int_0^1 2t^3 + 9t + 12t^5\, dt = 7$$

28. (Surface Area Review)
    (a) Let S be a surface with parametrization $\mathbf{r}(u, v)$, where $a \le u \le b$ and $c \le v \le d$. Write down the formula for the surface area of S.

(b) Now suppose S is the graph of a function $z = g(x, y)$. We can parametrize S by $\mathbf{r}(u, v) = (u, v, g(u, v))$. Using this parametrization, compute $||\mathbf{r}_u \times \mathbf{r}_v||$.

(c) Rewrite the formula of the surface area of S using the computation in part (b).

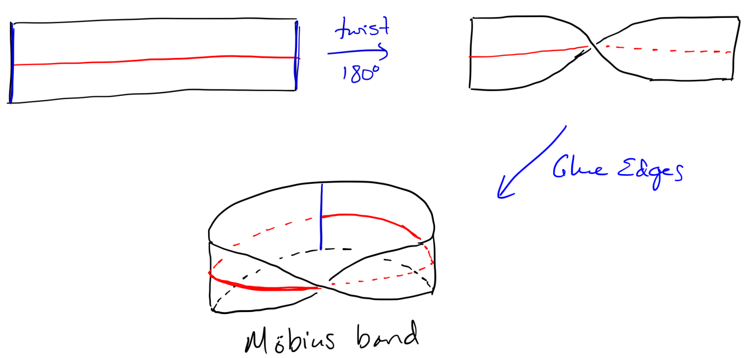
29. (An art project) Find a long rectangular strip of paper (cut the bottom 2 inches from a regular 8×11 sheet of paper, for example). Draw a straight line down the center of the strip (that is parallel to the long edge). If you glue the ends of the strip together without twisting it, you will have a short cylinder and the line you drew will be on the same side of the strip.



Instead, twist the strip 180°, as depicted below.

Now glue/staple the ends together. You are left with what is called a Möbius band. What happened to the line you drew? Is it still on the same side of the strip?

Solution: The line doesn't connect to itself and the strip only has one side.



30. (Calculus III Review) Let $\mathbf{a}$ and $\mathbf{b}$ be vectors and let θ be the acute angle between them. Recall the formula from Calculus III that relates the dot product of these two vectors with θ. Use the forumula to compute the angle between $(1, 0, 1)$ and $(5, 0, 0)$.

Solution: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$. Thus we have $5 = \sqrt{2}(5)\cos\theta$ and so $\theta = \frac{\pi}{4}$.

31. (Review of Orientation) Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, with the downward/inward orientation. Parametrize S.

Solution: If we parametrize by $\mathbf{r}(u, v) = (u, v, 4 - u^2 - v^2)$, then $\mathbf{r}_u \times \mathbf{r}_v = (2u, 2v, 1)$. When $u = 0$ and $v = 0$, we are at the point $\mathbf{r}(0, 0) = (0, 0, 4)$. At this point, the normal vector is $\mathbf{r}_u \times \mathbf{r}_v = (0, 0, 1)$, which points in the upward direction. Thus, we need the opposite orientation. To achieve this, we simply switch u and v: $\mathbf{r}(u, v) = (v, u, 4-v^2-u^2), 0 \leq u \leq 1, 0 \leq v \leq 1$.

32. (Review of Divergence and Curl). Let $\mathbf{F} = x\mathbf{i} - 2xyz\mathbf{j} + z^2x\mathbf{k}$. Compute div(curl$\mathbf{F}$).

Solution: Since $\mathbf{F}$ is a C^2 vector field, div(curl$\mathbf{F}$) $= 0$

33. (Review of Last Class)

    (a) Let S be the surface given by $x^2 + y^2 + z^2 = 1$, where $z \geq 0$, oriented inward. Notice that the boundary of S is the unit circle in the $xy-$plane. Is the orientation induced on ∂S clockwise or counterclockwise, when viewed from above? Parametrize ∂S.

    Solution: Since S is oriented inward, the induced orientation on ∂S is clockwise, when viewed from above. Thus we can parametrize it by $\mathbf{c}(t) = (\cos t, -\sin t, 0)$, $0 \leq t \leq 2\pi$.

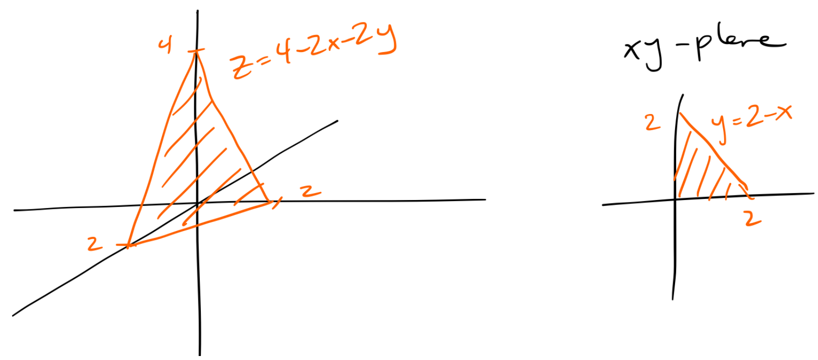
    (b) Use Stokes' Theorem to calculate $\displaystyle\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - 2\mathbf{j} + e^{xyz}\mathbf{j}$.

    Solution: $\displaystyle\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^{2\pi} \sin^2 t + 2\cos t\, dt$

    $\displaystyle= \int_0^{2\pi} (\frac{1}{2} - \frac{1}{2}\cos(2t)) + 2\cos t\, dt = \pi$

34. (Review of Triple Integrals) Evaluate $\displaystyle\iiint_W x\, dV$, where W is the solid region bounded by the surfaces $x = 0, y = 0, z = 0$, and $2x + 2y + z = 4$.

Solution: Consider the sketch of W below.



Based on the sketch, $W = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 2-x, 0 \le z \le 4-2x-2y\}$. Thus

$$\iiint\limits_W x\, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} x\, dz\, dy\, dx = \frac{4}{3}$$

35. (Calculus I, II, and/or III Review) Recall the *Mean Value Theorem for Integrals.* There is a version for single, double, and triple integrals.

- If f is continuous on $[a, b]$, then there exists a number x_0 in $[a, b]$ such that

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

- If f is continuous on $D \subset \mathbb{R}^2$, then there exists a point (x_0, y_0) in D such that

$$f(x_0, y_0) = \frac{1}{\text{Area}(D)} \iint\limits_D f(x, y)\, dA$$

- If f is continuous on $W \subset \mathbb{R}^3$, then there exists a point (x_0, y_0, z_0) in W such that

$$f(x_0, y_0, z_0) = \frac{1}{\text{Vol}(W)} \iiint\limits_W f(x, y, z)\, dV$$

Let $f(x) = 1 + x^2$ on $[-1, 2]$. Find a number x_0 in $[-1, 2]$ that satisfies the (first version) of the Mean Value Theorem.

Solution: Since $f(x_0) = 1 + x_0^2$ and $\dfrac{1}{2-(-1)} \displaystyle\int_{-1}^2 1 + x^2\, dx = 2$, by the Mean Value Theorem, $1 + x_0^2 = 2$ and so $x_0 = \pm 1$.

36. (Calculus I Review) Recall, that the differential of a function $y = f(x)$ is given by $dy = f'(x)dx$. More generally, the differential of a function of n variables $z = f(x_1, \ldots, x_n)$ is given by $dz = \dfrac{\partial f}{\partial x_1}\, dx_1 + \cdots + \dfrac{\partial f}{\partial x_n}\, dx_n$.

(a) Let $f(x) = x^2 + x$. Compute the differential of f and use it to estimate the change in f as x changes from 1 to 1.01.

Solution: $dy = (2x + 1)dx$. As x changes from 1 to 1.01, $dx = .01$ and so the change in f is approximately $dy = (2(1) + 1)(.01) = .03$.

(b) Let $z = f(x, y) = xy$. Compute the differential of f and use it to estimate the change in f as (x, y) changes from $(1, 1)$ to $(1.01, .08)$.

Solution: $dz = y\,dx + x\,dy$. As (x, y) changes from $(1, 1)$ to $(1.01, .08)$, we have that $dx = .01$ and $dy = -.02$. Thus the change in f is approximately $dz = (1)(.01)+(1)(-.02) = -.01$.

37. (Review of Last Class) Let $\alpha = 2xz\,dx + 3xy\,dy + 2e^z\,dz$ and $\beta = 3x\,dy - 2\,dz$ be 1-forms on $\mathbb{R}^3$. Compute $\alpha \wedge \beta$.

Solution:

$$\begin{aligned}
\alpha \wedge \beta &= (2xz\,dx + 3xy\,dy + 2e^z\,dz) \wedge (3x\,dy - 2\,dz) \\
&= 6x^2z\,dx\,dy - 4xz\,dx\,dz + 9x^2y\,dy\,dy - 6xy\,dy\,dz + 6xe^z\,dz\,dy - 4e^z\,dz\,dz \\
&= 6x^2z\,dx\,dy - 4xz\,dx\,dz - 6xy\,dy\,dz + 6xe^z\,dz\,dy \\
&= 6x^2z\,dx\,dy - 4xz\,dx\,dz - 6xy\,dy\,dz - 6xe^z\,dy\,dz \\
&= 6x^2z\,dx\,dy - 4xz\,dx\,dz - (6xy + 6xe^z)\,dy\,dz
\end{aligned}$$

38. (Review of Change of Variables) Use the change of variables $u = 2x + 3y$ and $v = x - y$ to rewrite the integrand of $\iint_R (2x + 3y)\sin(x - y)\,dx\,dy$. Do not forget about the determinant of the Jacobian.

39. (Review of Last Class) Let $\omega = A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$ be a 2-form on $\mathbb{R}^3$. Let $\mathbf{F}(x, y, z) = (A, B, C)$ be a vector field on $\mathbb{R}^3$. Show that $d\omega = \text{div}\mathbf{F}\,dx\,dy\,dz$.