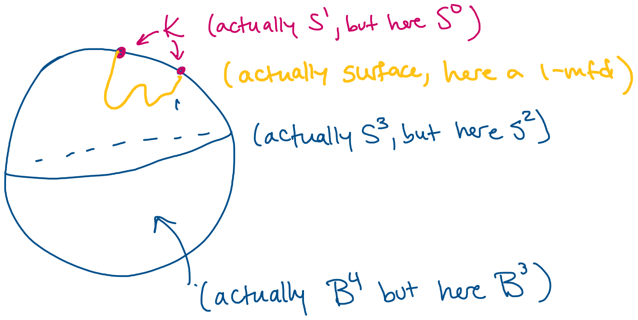
Ribbon surfaces

Monday, May 23, 2022 2:18 PM

Knots, surfaces and B^4



$K \hookrightarrow S^3 \cong \partial B^4$

goal: find interesting surfaces for K whose interiors lie in the interior of B^4

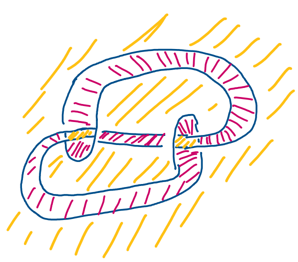
non-interesting example: take a Seifert surface for K and "push into B^4."

Q: How to push into B^4?

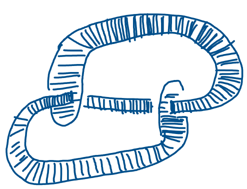
A: make interior "hotter."

Why are these not interesting? They don't rely on being in B^4 at all. We could have been in S^3.

Ribbon surfaces



checkerboard surface
Seifert's algorithm
$\hookrightarrow S^3$

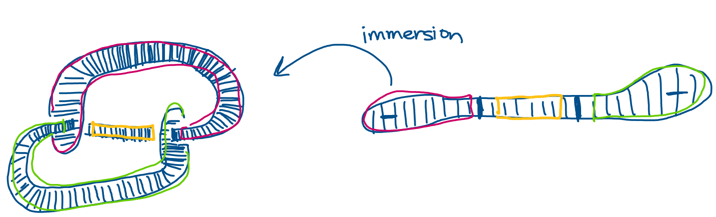


ribbon surface
$\hookrightarrow B^4$

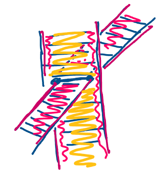
A ribbon surface in S^3 is a surface $S \hookrightarrow S^3$ (not an embedding) so that locally S looks like a surface OR we see a ribbon intersection:

i.e. in a neighborhood of the intersection we see →

on one "sheet" or band a properly embedded interval is in the intersection but in the other sheet just some interval in the interior is the intersection.

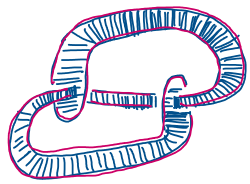


Q: Why ribbon singularities?
A: ribbon singularities in S^3 can be "resolved" in B^4



Have enough room (dimension) in B^4 to remove this intersection

Ribbon disk in B^4



bounded by $K \hookrightarrow S^3$
$\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;= 6_1$

here's a disk which is not ribbon immersed

This knot bounds a embedded disk in B^4, but not an embedded disk in S^3.

Def: the four-genus $g_4(K)$ for a knot $K \hookrightarrow S^3$ is the minimal genus of an orientable surface S properly embedded in B^4 with $\partial S = K$.

Ex: ① $g_4(6_1) = 0$ while $g_3(6_1) = 1$ (HW)
② $g_4(3_1) \leq 1$ and $g_3(3_1) = 1$
why?
(actually $g_4(3_1) = 1$)

Def A knot with $g_4(K) = 0$ or equivalently, a knot which bounds an embedded disk in B^4 is called slice.

If K bounds a ribbon immersed disk in S^3, it is called ribbon.

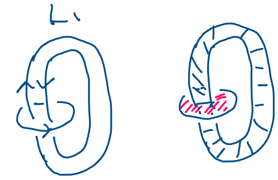
Conj: slice $\Rightarrow$ ribbon

since we can push a ribbon surface into B^4

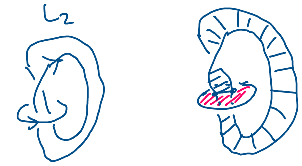
Links and ribbon surfaces

links can also bound ribbon surfaces: eg

L_1



L_2

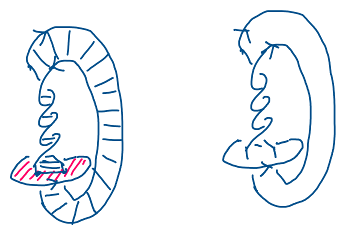


- L_1 bounds a disk union an annulus
- L_2 bounds a mobius band union a disk

☆ We don't ask that they be connected (just no closed components)

Instead of minimizing genus we want to maximize Euler characteristic.

☆ Again orientations matter



doesn't bound the disk union annulus that we see with the other or'n

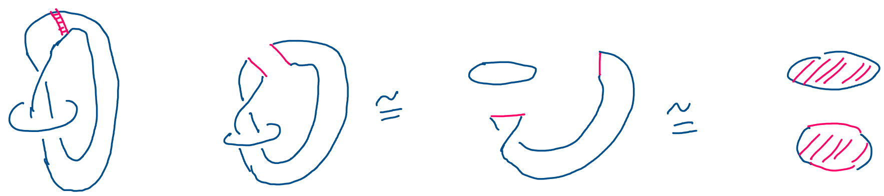
✯ Ribbon surfaces with Euler char. = 1 will play a special role for us.

- exs include a ribbon disk, and union of a disk and annulus|möbius band we saw

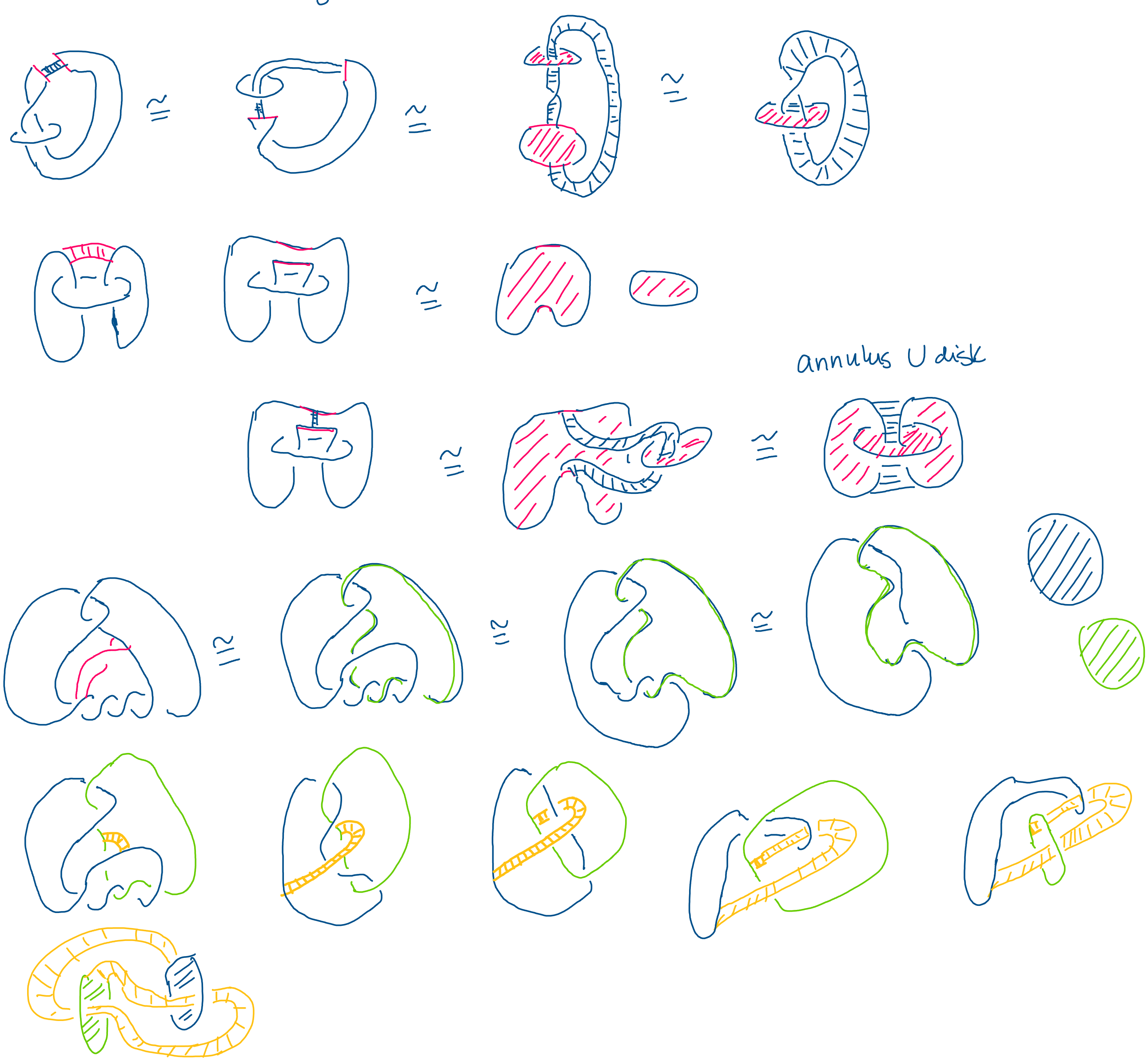
Def A link with a spanning surface S with $\chi(S)=1$ is called χ-slice.

✯ Note that slice knots are naturally χ-slice.

How to find ribbon surfaces?

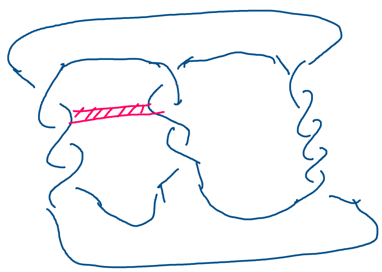


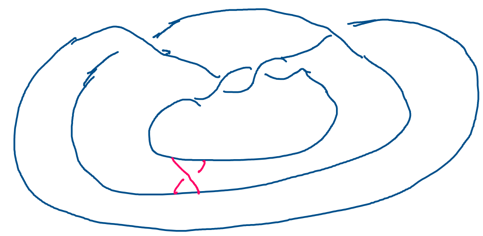
Add "ribbon" bands until get to the unlink (keeping track of "band needed to go back")



Problems

① Use the suggested band moves, to produce a ribbon surface for each knot. What surfaces (up to homeo) do you get? What Euler characteristics.

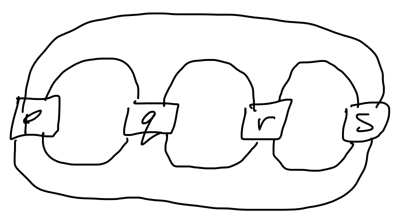




② Read prop 3.6 of Greene-Jabuka's "The slice-ribbon conjecture for 3-strand pretzel knots"

③ Let $p=-q$, $s \in \{-(r \pm 4), -(r \pm 1)\}$

Show that



is χ-slice