X-Slice links

Jon Simone
Georgia Tech
Knots: Knotted circles in $\mathbb{R}^3$

- $\bigcirc$ (unknot)
- $\bigcirc \bowtie$ (trefoil)
- $\bigcirc \bowtie \bowtie$ (figure-eight)

Links: Collection of knots in $\mathbb{R}^3$

Surfaces:

- $\bigcirc$ sphere
- $\bigcirc \bowtie$ torus
- $\bigcirc \bowtie \bowtie$ genus 2 surface
- $\bigcirc \bowtie \bowtie \bowtie$ genus $g$ surface
- $\circlearrowleft$ disk
- $\bigcirc \bowtie \bowtie \bowtie$ Mobius band
- etc.

(In topology, everything is made of rubber)

Note: The boundary of a surface in $\mathbb{R}^3$ is a link in $\mathbb{R}^3$
Fact: Every link in $\mathbb{R}^3$ is the boundary of an embedded surface in $\mathbb{R}^3$ (proved by Frankl-Pontryagin in 1930 and an algorithm to construct such surfaces was given by Seifert in 1934).

Surface doesn't intersect itself.

Valid surface

Not a valid surface

The only knot in $\mathbb{R}^3$ that bounds an embedded disk in $\mathbb{R}^3$ is the unknot.
Motivating Example

Let $K$ be a knot in $\mathbb{R}^2$ (it must be the unknot)

Let $D$ be a disk in $\mathbb{R}^2$ bounded by $K$

Then $D$ can be pushed into $\mathbb{R}^3$ giving a disk in $\mathbb{R}^3$ bounded by $K$, which is still in $\mathbb{R}^2$

Moreover, $K$ can bound other kinds of surfaces in $\mathbb{R}^3$ that it cannot bound in $\mathbb{R}^2$

The same is true for knots & links in $\mathbb{R}^3$:

Any surface bounded by a link $L$ in $\mathbb{R}^3$ can be pushed into $\mathbb{R}^4$

Moreover, $L$ can potentially bound more kinds of surfaces in $\mathbb{R}^4$ than in $\mathbb{R}^3$
Def: A knot in $\mathbb{R}^3$ is called slice if it bounds an embedded disk in $\mathbb{R}^4$.

Why a disk?
The simplest surface a knot can bound is a disk.
It is hard to bound simple surfaces
It is easy to bound more complicated surfaces.

Ex: The unknot is slice since it bounds a disk in $\mathbb{R}^3$ that we can push into $\mathbb{R}^4$

Ex: is slice even though it does not bound a disk in $\mathbb{R}^3$ (since it is not the unknot).

--- how can we tell?
Band Moves

Pick two arcs on $K$. Draw two arcs between the endpoints of the arcs you drew on $K$. Erase the arcs on $K$.

Fact: If we perform one band move to $K$ and get \(000\) then $K$ is slice.

Why?

These unknots band disjoint disks in $\mathbb{R}^4$.

Redraw the arcs that were erased to recover $K$. Fill in the arcs with a rectangle (band).

The result is a disk in $\mathbb{R}^4$ with boundary $K$.

More generally, if we perform $n$ band moves and get $0\ldots\ldots\ldots\ldots0$ then $K$ is slice.
Ex: A slice knot

We can also see the disk (made of 2 disks and a band)

This disk is in \( \mathbb{R}^3 \), but it intersects itself in \( \mathbb{R}^3 \), so it is an invalid surface in \( \mathbb{R}^3 \)

Now disk is embedded in \( \mathbb{R}^4 \) (i.e., it does not intersect itself in \( \mathbb{R}^4 \))

So \( K \) is slice.
Def: A link $L$ in $\mathbb{R}^3$ is slice if each component of $L$ bounds an embedded disk in $\mathbb{R}^4$ and the disks are disjoint.

Ex: $\quad \sim \Rightarrow \quad \circ \circ \quad = \quad \circ \circ$

This is a "classical" notion of sliceness for links that many researchers have studied.
Def (2012): A link \( L \) in \( \mathbb{R}^3 \) is \textcolor{red}{\textit{x-slice}} if \( L \) bounds an embedded surface in \( \mathbb{R}^4 \) with no closed components and Euler characteristic:

\[
\chi(S) = \# \text{vertices} - \# \text{edges} + \# \text{faces}
\]

\[ \chi = 4-4+1 = 1 \]

\[ \chi = 4-6+2 = 0 \]

Note: If \( L \) has one component, then \( x \)-slice = slice

(since the only \( x=1 \) surface w/ one boundary component is the disk)
**Why look at X-slice?**

- As with knots, the simplest surfaces that certain links can bound are those with $X=1$.
- X-slice links are also useful for constructing 3-dimensional and 4-dimensional objects that topologists are interested in studying.

To show a link is X-slice:

Use band moves as with knots.

We can also see the $X=1$ surface in this example:

$L$ bounds embedded Disk U Mobius band

$\Rightarrow X = 1 + 0 = 1$

$\Rightarrow L$ is X-slice
Summer REU: Determine which pretzel links are \( x \)-slice.

With:
- Hannah Turner (co-mentor)
- Weizhe Shen (grad TA)

Students:
- Sophia Fanelle (Barnard)
- Ben Heinemann (Utah)
- Evan Huang (Rice)

\[ P(-3,2,1) \quad P(2,-2,4,3) \]

Ex: \( P(x,-x,y) \) is \( x \)-slice
\[
\begin{align*}
\L & \quad \rightarrow \\
\L \quad & = 0 \\
\end{align*}
\]

Much is known about which pretzel knots are slice.

The goal of the REU was to extend these results to links and understand which pretzel links are \( x \)-slice.
Tools

- **Construction**
  Show certain infinite families of pretzel links are $x$-slice by using band moves

- **Obstruction** (this is the hard part)
  Show "most" pretzel links are not $x$-slice
  using algebraic tools

  e.g. Donaldson's Diagonalization Theorem, Lattice Embeddings, Heegaard Floer Homology $d$-invariants

  more or less understandable if you know linear algebra

(The majority of the summer was spent)
understanding and applying these obstructive tools
Some Results

- If \( p_1, p_2, \ldots, p_k > 0 \), then \( P(p_1, \ldots, p_k) \) is \( \alpha \)-slice if and only if
  \[
  (p_1, p_2, \ldots, p_k) = (k-3, 1, \ldots, 1) \text{ or } (p_1, p_2, \ldots, p_k) = (m+1, 1, \ldots, 1)
  \]

- The following 4-stranded pretzel links are \( \alpha \)-slice:
  \[
  P(p, q, -2, -2), \ P(p, q, -q, -p-1), \ P(p, q, -q, -p-4) \text{ (there are more)}
  \]

- If \( p, q, r, s \) satisfy some restrictive algebraic conditions, then \( P(p, q, r, s) \) is not \( \alpha \)-slice
  E.g. If \( p, q, r \geq 3 \) and \( r \leq -1 \), then \( P(p, q, r, s) \) is not \( \alpha \)-slice

Open Problem: finish the classification of \( \alpha \)-slice 4-stranded pretzel links

(slice 4-stranded knots were classified by Lecuona in 2013)
Thanks!

Jon Simone
jsimone7@gatech.edu

Challenge:
All links on this page are X-slice. Can you find band moves to prove it?
Answers:

1. \[ \begin{array}{c}
   \text{Diagram 1} \\
   \rightarrow \\
   \text{Diagram 2} \\
   \rightarrow \\
   \text{Diagram 3} \\
   \rightarrow \\
   \text{Diagram 4} \\
   \rightarrow \\
   \text{Diagram 5} \\
   \rightarrow \\
   \text{Diagram 6} \\
   \rightarrow \\
   \text{Diagram 7} \\
\end{array} \]