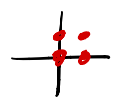
Cubiquity

Def: A sublattice $\Lambda \subset \mathbb{Z}^n$ is called <u>cubiquitous</u> if Λ contains a point in every unit cube with integer vertices.
A lattice embedding φ is called cubiquitous if $\operatorname{Im}\varphi$ is cubiquitous.

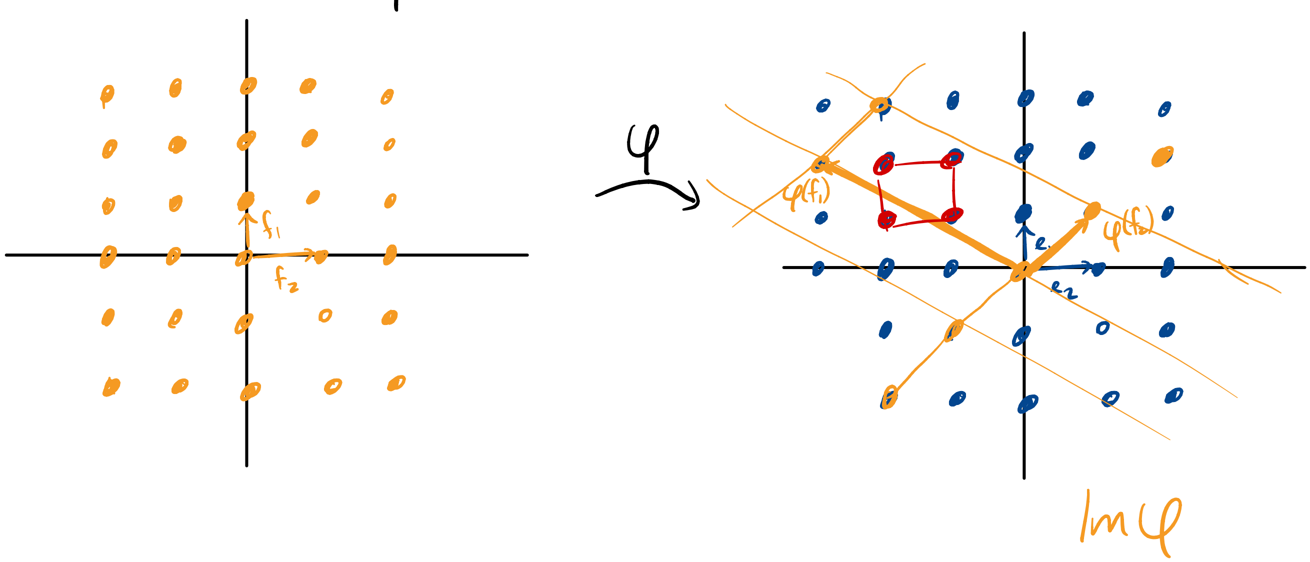
<u>Notation</u>: $\{0,1\}^n$ = unit cube with vertices having coordinates in $\{0,1\}$

$\{0,1\}^2$ = 

Ex: Let $Q = \begin{bmatrix} -13 & 1 \\ 1 & -2 \end{bmatrix}$

We have lattice embedding $\varphi : (\mathbb{Z}^2, Q) \to (\mathbb{Z}^2, -I)$

given by $\varphi(f_1) = -3e_1 + 2e_2$ (Check)
$\varphi(f_2) = e_1 + e_2$



This is <u>not</u> cubiquitous since the unit cube $C = \begin{bmatrix}-1\\1\end{bmatrix} + \{0,1\}^2 = \{(-1,1), (-1,2), (-2,1), (-2,-2)\}$ does not intersect $\operatorname{Im}\varphi$.

We can also prove this algebraically.

Show that $\forall \begin{bmatrix} a \\ b \end{bmatrix} \in C$, $\begin{bmatrix} a \\ b \end{bmatrix}$ is not a linear combination of $\varphi(f_1) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\varphi(f_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Leftrightarrow$ Show that $\nexists\, x, y \in \mathbb{Z}$ such that $x\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\forall \begin{bmatrix} a \\ b \end{bmatrix} \in C$

$\Leftrightarrow$ Show $\begin{matrix} -3x + y = a \\ 2x + y = b \end{matrix}$ has no integer solution $\forall \begin{bmatrix} a \\ b \end{bmatrix} \in C$

Ex: Recall, for $Q = \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix}$, we have a lattice embedding $\varphi : (\mathbb{Z}^2, Q) \to (\mathbb{Z}^2, -I)$ given by

$$\varphi(f_1) = -2e_1 + e_2$$
$$\varphi(f_2) = e_1 + e_2$$

We can show this embedding is cubiquitous.

Let C be an arbitrary unit cube in $\mathbb{Z}^n$.

Then $C = \begin{bmatrix} a \\ b \end{bmatrix} + \{0,1\}^n = \left\{ \begin{bmatrix} a + d_1 \\ b + d_2 \end{bmatrix} \,\middle|\, d_1, d_2 \in \{0,1\} \right\}$

for some integers a and b.

We need to show that $C \cap \operatorname{Im}\varphi \neq \emptyset$.

Equivalently, we need to show that

$\exists\, d_1, d_2 \in \{0,1\}$ such that $\begin{bmatrix} a + d_1 \\ b + d_2 \end{bmatrix} \in \operatorname{Im}\varphi$

Equivalently, we need to show that

$\exists\, \lambda_1, \lambda_2 \in \{0,1\}$ such that

$$\begin{aligned} -2x + y &= a + \lambda_1 \\ x + y &= b + \lambda_2 \end{aligned}$$ has an integer solution

Solving over $\mathbb{R}$ gives:

$$\left[\begin{array}{cc|c} -2 & 1 & a+\lambda_1 \\ 1 & 1 & b+\lambda_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & b+\lambda_2 - \frac{a+2b+\lambda_1+2\lambda_2}{3} \\ 0 & 1 & \frac{a+2b+\lambda_1+2\lambda_2}{3} \end{array}\right]$$

$$\Rightarrow \quad x = b + \lambda_2 - \frac{a+2b+\lambda_1+2\lambda_2}{3}, \quad y = \frac{a+2b+\lambda_1+2\lambda_2}{3}$$

We have 3 cases to consider: $a+2b \equiv 0,1,2 \mod 3$

- If $a+2b \equiv 0 \mod 3$, then $3 \mid a+2b$

  Let $\lambda_2 = \lambda_1 = 0$. Then $y = \frac{a+2b}{3} \in \mathbb{Z}$ and $x = b - \frac{a+2b}{3} \in \mathbb{Z}$
- If $a+2b \equiv 1 \equiv -2 \mod 3$, then $3 \mid a+2b+2$

  Let $\lambda_1 = 0$ and $\lambda_2 = 1$. Then $y = \frac{a+2b+2}{3} \in \mathbb{Z}$ and $x = b+1 - \frac{a+2b+2}{3} \in \mathbb{Z}$
- If $a+2b \equiv 2 \equiv -1 \mod 3$, then $3 \mid a+2b-2$

  Let $\lambda_1 = 1$ and $\lambda_2 = 0$. Then $y = \frac{a+2b+1}{3} \in \mathbb{Z}$ and $x = b - \frac{a+2b+1}{3} \in \mathbb{Z}$

Hence, in every case, $\exists\, \lambda_1, \lambda_2 \in \{0,1\}$ such that the system has an integer solution.

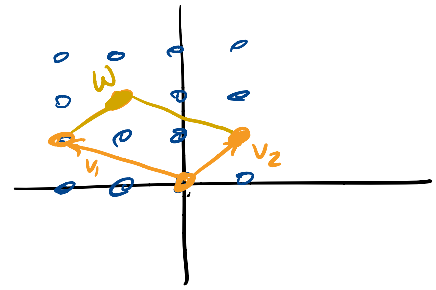
$\Rightarrow \operatorname{Im}\varphi \cap C \neq \emptyset$.

Working Geometrically

Def: Let $S=\{v_1,\dots,v_n\}\subset\mathbb{Z}^n$ be a subset of vectors (e.g. $v_i=\varphi(f_i)$ for some lattice embedding φ). The *Wu element* of S is $W=\sum_{i=1}^{n} v_i$

Ex: $S=\{-2e_1+e_2,\ e_1+e_2\}\subset\mathbb{Z}^2$

$W=-2e_1+e_2+e_1+e_2=-e_1+2e_2$



Note: Geometrically, if D is the fundamental parallelopiped spanned by S, then W is the corner opposite O.

Def: Let $S=\{v_1,\dots,v_n\}$ be a subset of $(\mathbb{Z}^n,-I)$.

Let $v_i\cdot v_i=a_i\geq 2\ \forall i$.

S is called:

standard if $v_i\cdot v_j=\begin{cases}1 & \text{if } |i-j|=1\\ 0 & \text{if } |i-j|>1\end{cases}$

+ *cyclic* if $a_j\geq 3$ for some j

and $v_i\cdot v_j=\begin{cases}1 & \text{if } |i-j|=1 \text{ or } i\neq j\in\{1,n\}\\ 0 & \text{else}\end{cases}$

Def: Given a subset $S = \{v_1, \dots, v_n\}$ of $(\mathbb{Z}^n, -I)$ with $v_i \cdot v_i = -a_i \ \forall i$, we define $I(S) := \sum_{i=1}^{n} (a_i - 3)$

Ex: $S = \{-2e_1 + e_2, e_1 + e_2\}$ is standard.

$v_1 \cdot v_1 = -5, \quad v_2 \cdot v_2 = -2, \quad v_1 \cdot v_2 = 1$

$I(S) = 2.$

Theorem: Let $S = \{v_1, \dots, v_n\}$ be a standard or +cyclic subset with $I(S) > 0$

Let $W = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} k_i e_i$ be its Wu element.

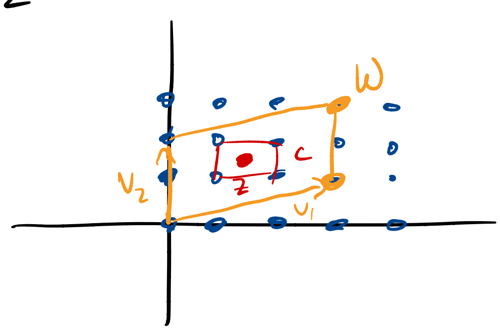
If k_i is odd for all i then the lattice Λ spanned by S is not cubiquitous.

proof (for +cyclic case)

let $z = \frac{1}{2} W$ (centroid of $D = \text{Span}\, S$).

Then $z = \sum_{i=1}^{n} \frac{k_i}{2} e_i$. Note that $\frac{k_i}{2} \notin \mathbb{Z}$ since k_i is odd

Let C be the unit cube with centroid z



Then $\forall y \in C, \ d(y, z) = \frac{\sqrt{n}}{2}$

We will show $\forall x \in \Lambda, \ d(x, z) > \frac{\sqrt{n}}{2}$, which implies $\Lambda \cap C = \emptyset \Rightarrow \Lambda$ is not cubiquitous.

Let $x \in \Lambda$. Then $x = \sum_{i=1}^{n} \lambda_i v_i$.

$$d(x,z)^2 = \| x - z \|^2 = \left\| \sum_{i=1}^{n} \lambda_i v_i - \sum_{i=1}^{n} \tfrac{1}{2} v_i \right\|^2 = \left\| \sum_{i=1}^{n} (\lambda_i - \tfrac{1}{2}) v_i \right\|^2$$

$$= \left\langle \sum (\lambda_i - \tfrac{1}{2}) v_i , \sum (\lambda_i - \tfrac{1}{2}) v_i \right\rangle$$ (usual dot product)

$$= \sum_{i=1}^{n} (\lambda_i - \tfrac{1}{2})^2 \underbrace{\langle v_i, v_i \rangle}_{=a_i} + 2 \sum_{i=1}^{n} (\lambda_i - \tfrac{1}{2})(\lambda_{i+1} - \tfrac{1}{2}) \underbrace{\langle v_i, v_{i+1} \rangle}_{=-1}$$ $\leftarrow n+1 = 1$

$$= \sum_{i=1}^{n} \frac{(2\lambda_i - 1)^2}{4} a_i - \sum_{i=1}^{n} \frac{4\lambda_i\lambda_{i+1} - 2\lambda_i - 2\lambda_{i+1} + 1}{2}$$

$$= \frac{(2\lambda_1 - 1)^2}{4}(a_1 - 1) + \frac{(2\lambda_1 - 1)^2}{4} + \frac{(2\lambda_2 - 1)^2}{4}(a_2 - 1) + \frac{(2\lambda_2 - 1)^2}{4} + \sum_{i=3}^{n} \frac{(2\lambda_i - 1)^2}{4} a_i$$

$$- \frac{4\lambda_1\lambda_2 - 2\lambda_1 - 2\lambda_2 + 1}{2} - \sum_{i=2}^{n-1} \frac{4\lambda_i\lambda_{i+1} - 2\lambda_i - 2\lambda_{i+1} + 1}{2}$$

$$= \frac{(2\lambda_1 - 1)^2}{4}(a_1 - 1) + \frac{(2\lambda_2 - 1)^2}{4}(a_2 - 1) + \sum_{i=3}^{n} \frac{(2\lambda_i - 1)^2}{4} a_i$$

$$(\lambda_1 - \lambda_2)^2 - \sum_{i=2}^{n} \frac{4\lambda_i\lambda_{i+1} - 2\lambda_i - 2\lambda_{i+1} + 1}{4}$$

⋮ repeat for all terms in last sum

$$= \sum_{i=1}^{n} \frac{(2\lambda_i - 1)^2}{4}(a_i - 2) + \sum_{i=1}^{n} (\lambda_i - \lambda_{i+1})^2$$

$$\geq \sum_{i=1}^{n} \frac{a_i - 2}{4} = \frac{\sum (a_i - 3) + n}{4} = \frac{I(S) + n}{4} > \frac{n}{4}$$

$$\Rightarrow d(x,z) > \frac{\sqrt{n}}{2}.$$

$\blacksquare$

Ex: Let $Q = \begin{bmatrix} -5 & 1 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ and consider the

lattice embedding $\varphi : (\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, -I)$

given by

$$\varphi(f_1) = 2e_1 - e_2$$

$$\varphi(f_2) = e_2 - 3e_3$$

$$\varphi(f_3) = -e_1 - e_2$$

The subset $\{2e_1 - e_2, e_2 - 3e_3, -e_1 - e_2\}$ is + cyclic with $I(S) = 7 > 0$ and $W = e_1 - e_2 - 3e_3$

Thus by the theorem, φ is not cubiquitous.