d-invariants

Let Y be a $\mathbb{Q}S^3$.
Given a spin-c structure s (an element of $H^2(Y)$)
the d-invariant of (Y,s) is $d(Y,s) \in \mathbb{Q}$, which is defined
using "Heegaard Floer homology".

Thm 1 (Ozsvath-Szabo)
If Y bounds a $\mathbb{Q}B^4$, B, and t is a spin-c str on B
then $d(Y, t|_Y) = 0$

restriction of t to Y

Thm 2 Let Y bound a $\mathbb{Q}B^4$, B.
then $|H^2(Y)| = m^2$ and $\exists$ subgroup $V \subset H^2(Y)$
with $|V| = m$ s.t. $d(Y,s) = 0 \;\; \forall s \in V$

Note $m^2 = |\det L|$ if Y is the DBC of a link L

V = Image of $H^2(W) \to H^2(Y)$

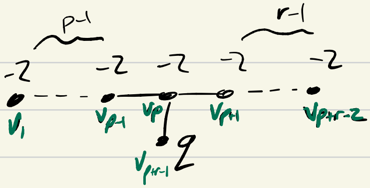
Let X be negative-definite 4-mfld w/ $\partial X = Y$
Suppose Y bounds $\mathbb{Q}B^4$ B
Then $\exists$ lattice embedding $\varphi : (H_2(X), Q_X) \longrightarrow (\mathbb{Z}^n, -I)$

$$(\mathbb{Z}^n, -I) \cong (H^2(X \cup B), Q_{X\cup B})$$

We can represent φ by a matrix A^T

Facts:
- We can choose bases for $H^2(X)$ & $H^2(X \cup B)$ s.t. the restriction map $H^2(X\cup B) \longrightarrow H^2(X)$ can be represented by A

$$\left.\begin{array}{l} \bullet\; Q = -AA^T \\ \bullet\; V \cong \operatorname{Im} A / \operatorname{Im} G \end{array}\right\} \text{ see Jabuka-Greene}$$

Ex: $X =$ 

Let $H_2(X)$ have basis given by 2-handles $\{v_1, \ldots, v_{p+r}\}$.
Let Q be its intersection form.
Choose basis for $H_2(X \cup B)$ $(\{e_1, \ldots, e_n\})$ for which its intersection form is $-I$ (possible by Donaldson's thm)
Then:

$$A^T = \begin{array}{c} \\ p \\ \\ r \\ \\ \end{array} \begin{bmatrix} \overbrace{\begin{matrix} 1 & & \\ -1 & 1 & \\ & -1 & \ddots \\ & & 1 \\ & & -1 \end{matrix}}^{p-1} & & \begin{matrix} \lambda \\ \lambda \\ \vdots \\ \\ \lambda \end{matrix} \\ \begin{matrix} 1 \\ -1 \end{matrix} & \underbrace{\begin{matrix} 1 & & \\ -1 & \ddots & \\ & & 1 \\ & & -1 \end{matrix}}_{r-1} & \begin{matrix} \lambda+1 \\ \vdots \\ \lambda+1 \end{matrix} \end{bmatrix} \qquad Q = \begin{bmatrix} -2 & 1 & & & & 1 & & 1 \\ 1 & -2 & \ddots & & & & & \\ & \ddots & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ 1 & & & 1 & \ddots & 1 & & \\ & & & & 1 & -2 & & \\ 1 & & & & & & & q \end{bmatrix}$$

Check: $Q = -AA^T$

Goal: Show $\#\{s \in Spin^c(Y) \mid s \in V \text{ and } d(Y,s) = 0\} < |V|$

Fact: $Spin^c(X \cup B) \approx$ "Characteristic elements" of $H^2(X \cup B)$
(i.e. vectors in $H^2(X \cup B) \cong \mathbb{Z}^n$ whose entries are all odd)

Let $\mathrm{Char}_s(X \cup B) = \{$ characteristic elements of $H^2(X \cup B)$ whose associated spin-c str on $X \cup B$ restricts to s on $Y \}$

Thm (Ozsváth-Szabó): Let X be a negative-definite plumbing whose graph is a tree with at most 2 bad vertices w/ $H_2(X) \cong \mathbb{Z}^n$.

Further assume $\partial X = Y$ bounds a $\mathbb{Q}B^4$, B. Then

$$d(Y, s) = \max_{x \in \mathrm{Char}_s(X \cup B)} \frac{n - x \cdot x}{4}$$

← regular dot product

Notice, $d(Y, s) = 0$ iff $x = (\pm 1, \pm 1, \ldots, \pm 1) \in \mathbb{Z}^n$

So every element of $\{(x_1, \ldots, x_n) \in H^2(X \cup B) \mid x_i = \pm 1\}$ is a representative of an equivalence class of characteristic elements in $H^2(X \cup B)$ restricting to the same spin-c str on Y and whose d-invariant is 0.

However, some of these elements might still belong to the same equivalence classes.

Recall, $V \cong \mathrm{Im}A/\mathrm{Im}G$

Claim: Let $v = Ax$, $v' = Ax' \in \mathrm{Im}A$

Then $[v] = [v'] \in V \iff x - x' \in \mathrm{Im}A^T$

proof:

$[v] = [v'] \in \mathrm{Im}A/\mathrm{Im}Q$

$\iff v - v' \in \mathrm{Im}Q$

$\iff v - v' = Qy$ for some y

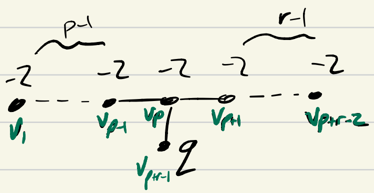
$\iff v - v' = -AA^Ty$

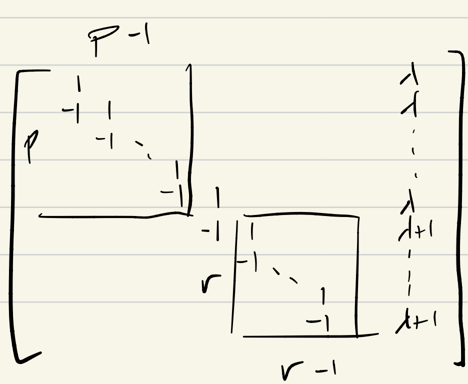
$\iff Ax - Ax' = -AA^Ty \iff A(x - x') = -AA^Ty$

$\iff x - x' = -A^Ty$ (since A is one-to-one)

$\iff x' - x = A^Ty \iff x' - x \in \mathrm{Im}A^T$ $\blacksquare$

Back to Jabuka-Greene Example

$X =$ 

$A^T =$ 

Recall, using lattice analysis, we know

if $\exists$ a lattice embedding, then $q = -p\lambda - r(\lambda+1)^2$

Claim (Jabuka-Greene): If $P(p,q,r)$ is slice, then $\lambda \in \{0,-1\}$.

proof:

Let $\ell : \mathbb{Z}^{p+r} \to \mathbb{Z}$,
$\ell(x_1, \ldots, x_{p+r}) = \Sigma x_i$

$\operatorname{Im} A^T$ = All linear combos of columns of A

Moreover, $\operatorname{Ker} \ell \cong \operatorname{Span}\{$1st $p+r-1$ columns of $A\}$

$\Rightarrow \operatorname{Ker} \ell \subset \operatorname{Im} A^T \cong \mathbb{Z}^{p+r}$

View $\ell : \operatorname{Im} A^T \longrightarrow \mathbb{Z}$

Thus if $v = Ax,\ v' = Ax' \in \operatorname{Im} A$ and $x - x' \in \operatorname{Ker} \ell \subset \operatorname{Im} A^T$,

then by previous claim, $[v] = [v'] \in V = \operatorname{Im} A / \operatorname{Im} Q$

That is, if $\ell(x) = \ell(x')$ (ie. sum of entries of x = sum of entries of x')

then $[v] = [v'] \in V$.

Now ℓ restricted to $\{(x_1,\ldots,x_n) \in H^2(X\cup B) \cong \mathbb{Z}^{p+r} \mid x_i = \pm 1\}$ takes on $p+r+1$ distinct values

(corresponding to the # of negative entries of x)

$\Rightarrow \exists$ at most $p+r+1$ spin-c strs in V s.t. $d=0$

On the other hand, by thm 2, $d(Y,s)=0 \ \forall s \in V$

$\Rightarrow |V| \leq p+r+1$

<u>Note</u>: $|V| = \sqrt{|\det P(p,q,r)|} = \sqrt{pq+qr+pr}$

$$= \sqrt{p(-p\lambda^2 - r(\lambda+1)^2) + r(-p\lambda^2 - r(\lambda+1)^2) + pr}$$

$$= |p\lambda + r(\lambda+1)|$$

$\Rightarrow |p\lambda + r(\lambda+1)| \leq p+r+1$

$\Rightarrow \lambda = 0, -1.$ ∎