Double Branched Covers

Let X be an n-manifold.

Let $B \subset X$ be an $(n-2)$-dimensional submanifold (branch locus)

A double cover of X branched along B is an n-manifold Z, along with a continuous map $f: Z \to X$ satisfying

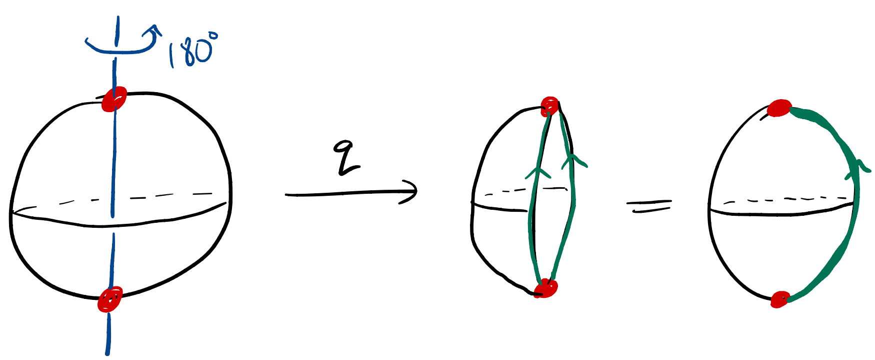
- $f^{-1}(B)$ is an $(n-2)$-dimensional submanifold of Z
- $f|_{Z - f^{-1}(B)} : Z - f^{-1}(B) \longrightarrow X - B$ is a double cover

Ex: S^2 is a double cover of S^2 branched along two points.

View $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ and $B = \{(0,0,1),(0,0,-1)\}$

Let $i: S^2 \to S^2$ be $180°$-rotation about z axis. Then $i(B) = B$ and the quotient of S^2 by i is $S^2/i \approx S^2$.

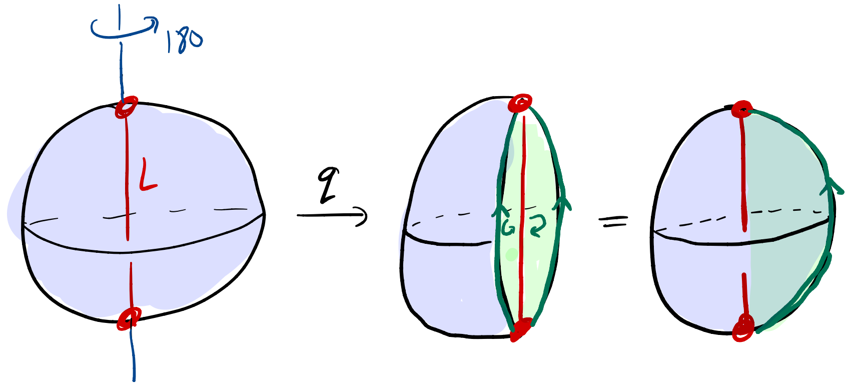
Let $q: S^2 \longrightarrow S^2/i \cong S^2$ denote the quotient map.



then $q|_{S^2 - \{x,y\}}$ is a double cover and $q^{-1}(B) = \{x,y\} = B$.

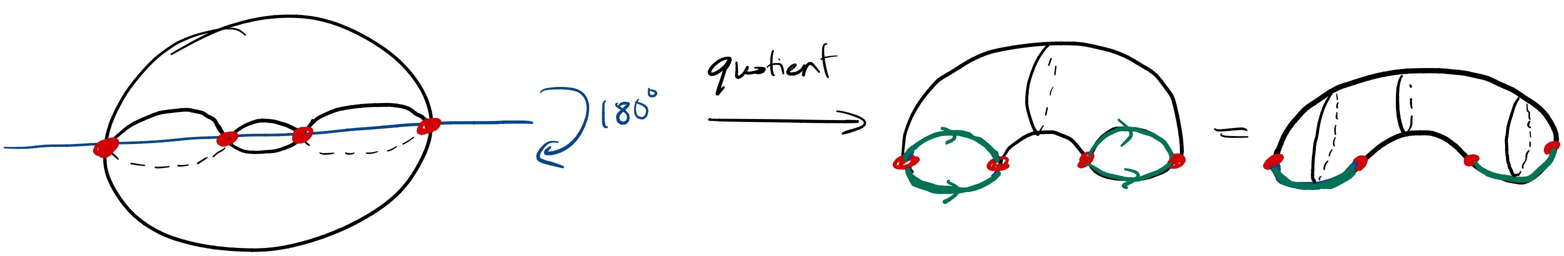
Ex: B^3 is a double branched cover of B^3 branched along a line segment L with endpoints on ∂B^3.

As above, let $i: B^3 \to B^3$ be given by $180°$-rotation about the z-axis. Notice, $B^3/i \approx B^3$ and $i(L)=L$. Let $q: B^3 \longrightarrow B^3/i \simeq B^3$ be the quotient map.

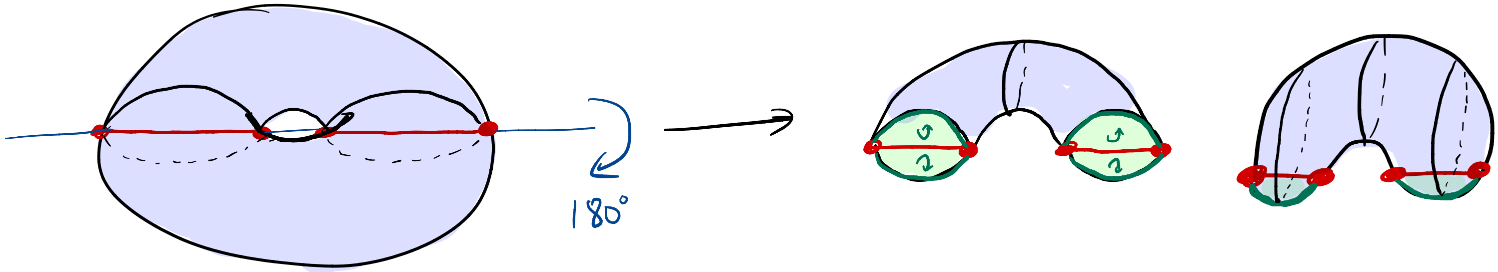


Then $q|_{B^3-B}$ is a double cover and $q^{-1}(B) = B$.

Ex: T^2 is a double cover of S^2 branched along 4 points



Ex: $S^1 \times D^2$ is a double cover of B^3 branched along two arcs with endpoints on $\partial B^3 = S^2$.



In general, if X has boundary and $B \subset X$ has $\partial B \subset \partial X$, then the boundary of a double cover of X branched along B is a double cover of ∂X branched along ∂B.

Increasing dimension, let's think about branched covers of S^3 and B^4.

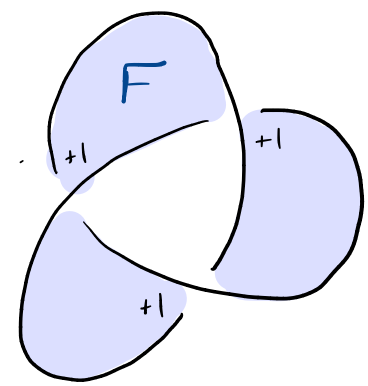
Ex: S^3 is a double cover of S^3 branched along the unknot $\mathcal{U}$. B^4 is a double cover of B^4 branched along a spanning disk of $\mathcal{U}$.

We are interested in the following situation. Suppose $L \subset S^3$ is a link that bounds a surface $F \subset B^4$ with no closed components.

It turns out $\exists$ unique double covers of S^3 and B^4 branched along L and F, respectively. We denote them by $\Sigma_2(S^3, L)$ and $\Sigma_2(B^4, F)$. Note that $\partial \Sigma_2(B^4, L) = \Sigma_2(\partial B^4, \partial L) = \Sigma_2(S^3, L)$
We would like to understand these manifolds.

An algorithm

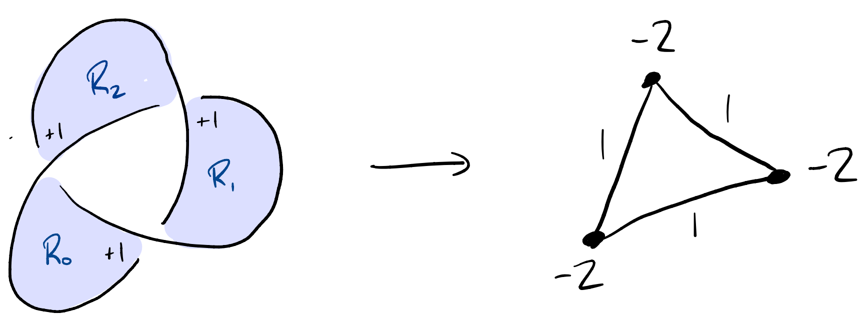
Let L be any link and choose a diagram of L.
Let F be a spanning surface for L
(not necessarily orientable)



Assign each crossing $+1$ or -1 according to the convention:

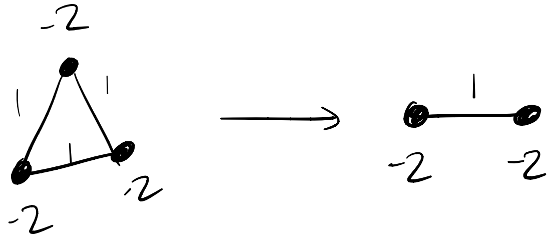
The Tait graph associated to F, denoted by Γ_F, is obtained as follows. Label the regions $R_0, \dots, R_n$.

- $R_i \rightsquigarrow$ vertex v_i with weight $-\sum$ signs of crossings incident to R_i
- ± 1 crossing between R_i and $R_j \rightsquigarrow \pm 1$ edge between v_i and v_j



We associate a symmetric bilinear form $\mathcal{G}_F$ on $H_1(F)$ called the Gordon-Litherland form as follows:
Delete R_0 and the edges incidence to R_0.

Let G_F be the incidence matrix of the resulting graph.



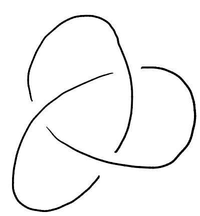
$\Rightarrow G_F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

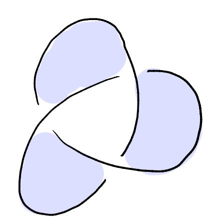
Then G_F represents $\mathcal{G}_F$ in some basis for $H_1(F)$.

We call G_F a Goeritz matrix for F

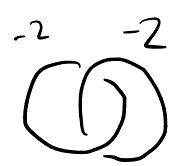
Facts • If F is orientable, then $V+V^T$ also represents $\mathcal{G}_F$, where V is a Seifert matrix V of L.

• G_F represents the intersection form of $\Sigma_2(B^4, F)$ (after F is pushed into B^4)

Ex: For the trefoil $K =$ 

and the spanning surface $F =$ Mobius band $=$ 

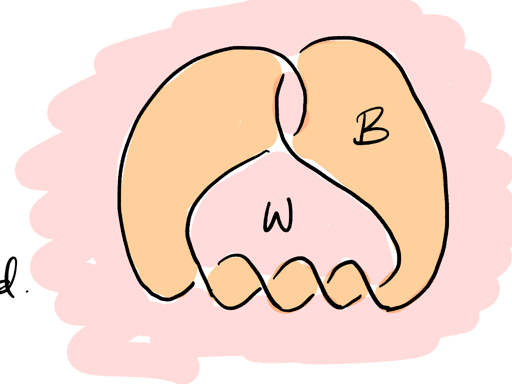
$\Sigma_2(S^3, K)$ bounds $\Sigma_2(B^4, F)$, which has intersection form $G_F = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

$\Sigma_2(B^4, F)$ is given by the handlebody diagram 

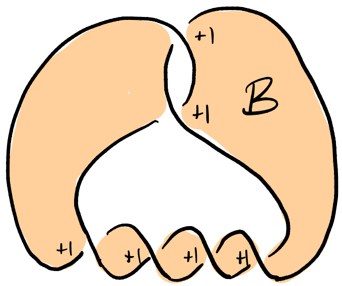
Moreover, $\Sigma_2(S^3, K)$ is a "lens space".

Using Checkerboard Surfaces

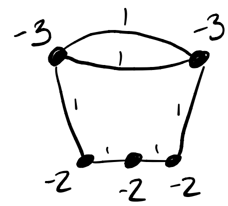
Given a fixed diagram for $L \subset S^3$, there are two <u>checkerboard surfaces</u> spanning L, which we call B and W with the convention that B is bounded.



Each gives a Goeritz matrix



$\rightarrow$

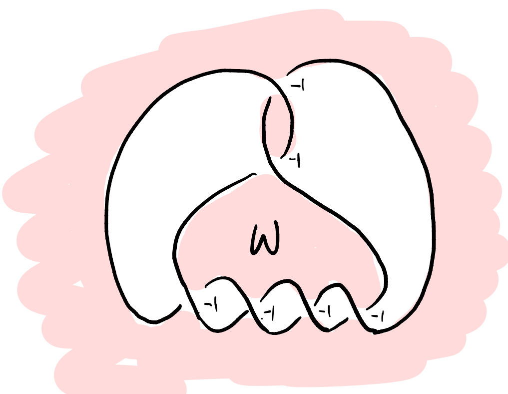


Tait graph

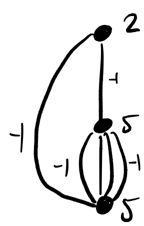
$\rightarrow$

$$\Rightarrow G_B = \begin{bmatrix} -3 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix}$$

negative definite

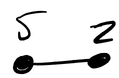


$\rightarrow$



Tait graph

$\longrightarrow$



$$\Rightarrow G_W = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

positive definite

Fact: If L is <u>alternating</u>, then G_B and G_W are definite.

L admits a diagram whose crossings alternate between over and under

In particular, one is positive definite and the other is negative definite.

So $\Sigma_2(B^4, B)$ and $\Sigma_2(B^4, W)$ are both definite
(one is positive definite, one is negative definite)

Note: For G_W and G_B in the last example,
$\exists$ lattice embeddings

$$(\mathbb{Z}^2, G_W) \longrightarrow (\mathbb{Z}^2, I) \quad \text{and} \quad (\mathbb{Z}^4, G_B) \longrightarrow (\mathbb{Z}^4, -I)$$

(You know $\exists$ lattice embeddings

$$(\mathbb{Z}^2, -G_W) \longrightarrow (\mathbb{Z}^2, -I) \quad \text{and} \quad (\mathbb{Z}^4, G_B) \longrightarrow (\mathbb{Z}^4, -I)$$

from an old homework)

This doesn't happen for all links. We'll see why this embedding exists next time.

(Hint: it exists because the knot we started with is slice).