Dehn Surgery

Dehn Surgery

Closed, connected, oriented 3-man.s:

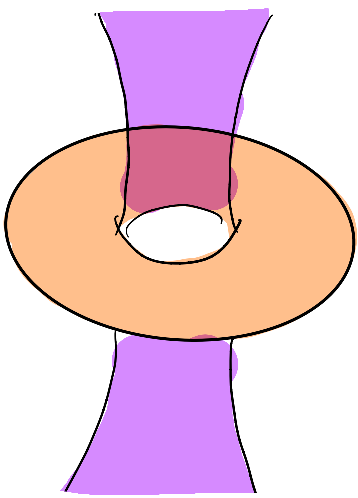
S^3, $S^1 \times S^2$, $L(p,q)$, others?

Goal: Construct every closed, oriented, connected 3-man. (using links in S^3)

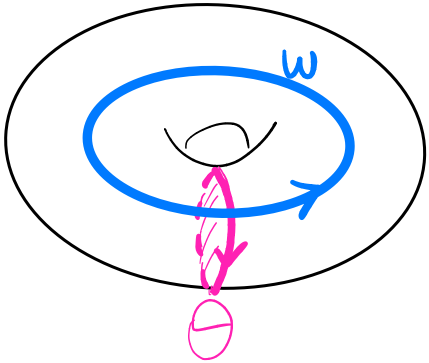
Examples:

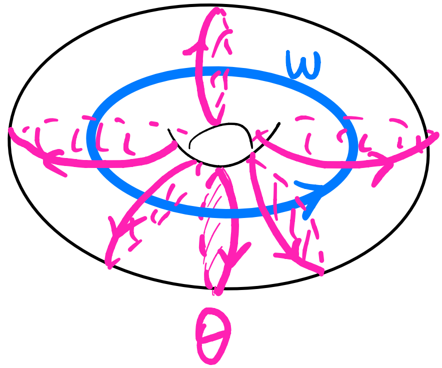
① $S^3 = \left[S^3 - \underbrace{(S^1 \times D^2)}_{\nu\mathcal{U}} \right] \cup_{id} \left[(S^1 \times D^2) \right]$

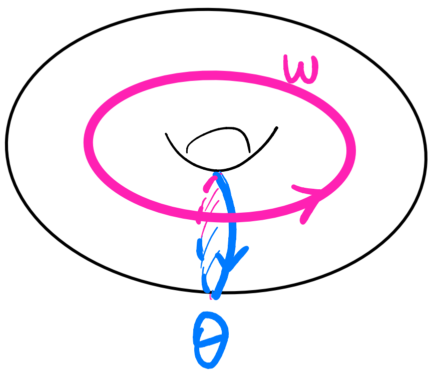
$S^1 \times D^2$

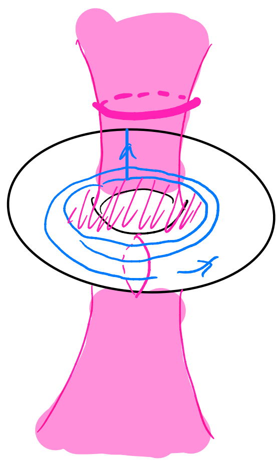


② $S^1 \times S^2$

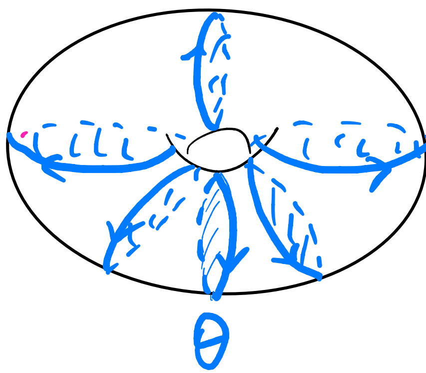








$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$T^2 \cong S^1 \times S^1$ parametrized by

$(\theta, \omega) \in \mathbb{R}^2$ $\qquad (\theta, \omega) \longmapsto (e^{2\pi i \theta}, e^{2\pi i \omega})$

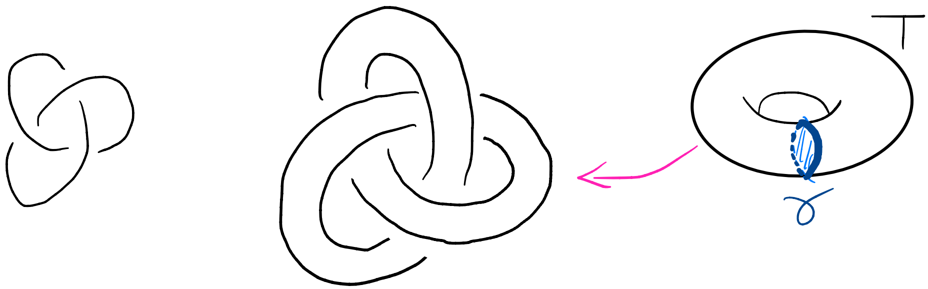
$$\varphi_{S^3} : T^2 \longrightarrow T^2$$
$$(\theta, \omega) \mapsto (\omega, -\theta)$$

Q: What manifolds do we get?

A: $S^1 \times S^2$

Non-trivial Dehn Surgery on unknot to obtain $S^1 \times S^2$

Surgery on Knots



need homeomorphism $\varphi : T^2 \to T^2$

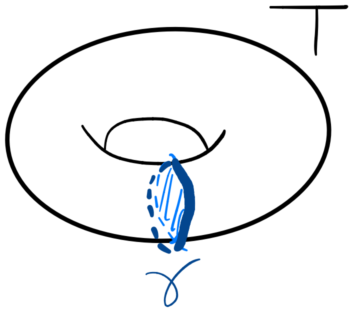
to get 3-man.

Q: What is φ ?

A: Actually, we don't need to know φ,

we just need to know

where γ goes. Why?



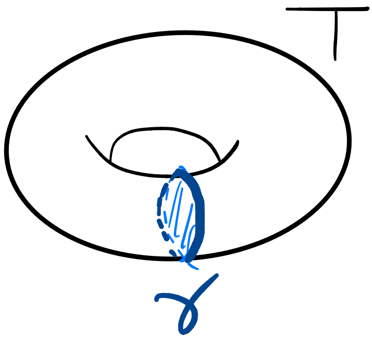
If we cut T along γ get a B^3.

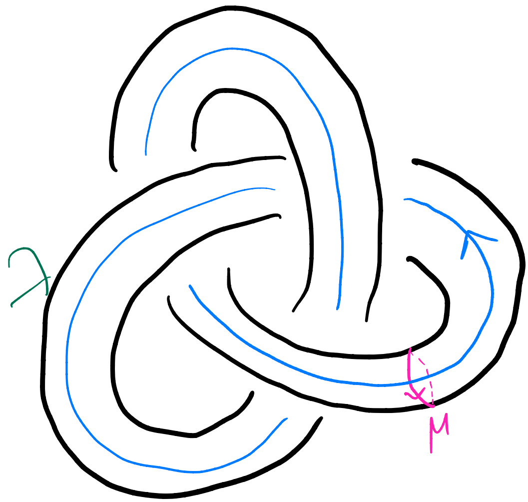
Once we know where φ takes γ,

this determines the gluing uniquely

since $\exists$ only one way to glue B^3 to S^2.

Q: What could $\varphi(\gamma)$ be?



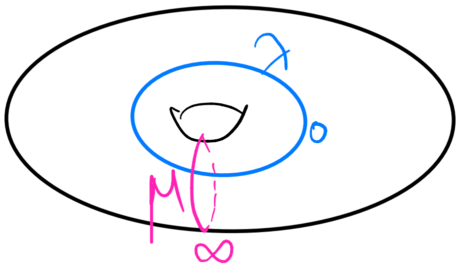


$\varphi(\gamma)$ could be any simple closed curve but up to isotopy, we can assume

$$\varphi(\gamma) = p\lambda + q\mu$$

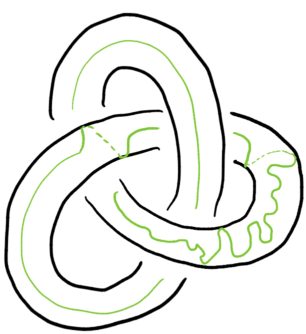
$\{\text{embd curves on } T^2\} \longleftrightarrow \mathbb{Q}\cup\{\infty\}$

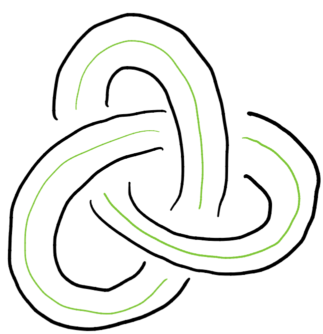
$\gamma \longleftrightarrow p/q$



$\gamma = p\lambda + q\mu$

$\gamma \xhookrightarrow[\text{embd}]{} T^2 \implies (p,q) = 1$





$= 0\lambda + 1\mu$

Since $p\lambda + q\mu$ is a simple closed curve,

we can assume $(p,q)=1$ and

p/q is a reduced fraction (or ∞).

The value p/q determined by a curve γ

is called it's slope.

A knot and a slope determine a surgery.

Dehn Surgery along K in S^3

K : a knot

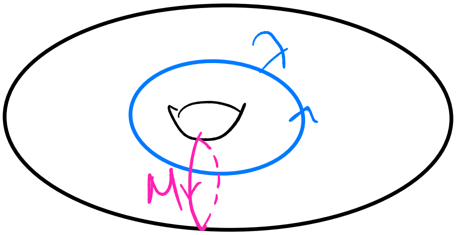
$\nu(K) :=$ tubular nbhd of K.

$$S^3_{p/q}(K) := \left(S^3 - \mathrm{Int}\,\nu(K)\right) \bigcup_{\varphi} \left(S^1 \times D^2\right)$$

$$\varphi : \partial(S^1 \times D^2) \longrightarrow \partial(\nu K)$$

Glue back in $\underbrace{\text{along } (p,q)\text{-curve}}_{\text{using a twist of } \partial(S^1 \times D^2)}$

Ex: In our first two example if λ and μ are as pictured:

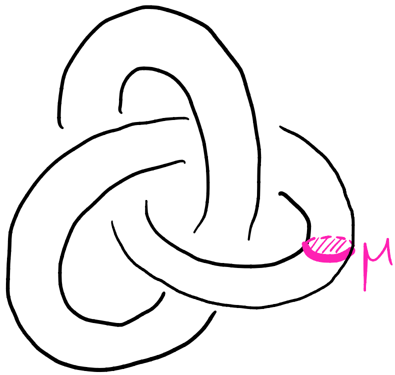


$S^3 - \nu(U)$

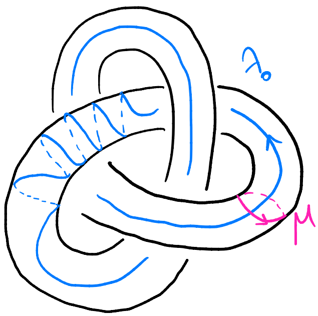
1. $S_{1/0}(U) = S^3$

2. $S_{0/1}(U) = S^1 \times S^2$

Chosing μ and λ carefully:



μ is always chosen to be $\partial D^2 \times \{pt\}$ in $\overline{\nu(K)}$



λ is chosen as $\#(\mu, \lambda) = 1$

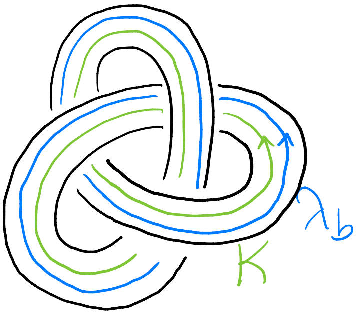
(all choices are NOT isotopic)

We need to be more careful to agree on choice of λ.

Choose λ so that $lk(\lambda, K) = 0$ in $\nu(K)$

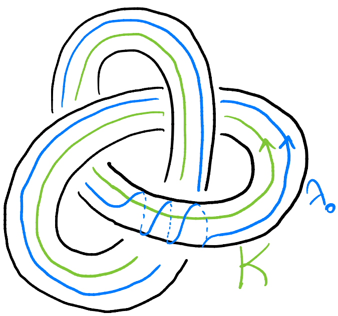
Ex:

①



$lk(\lambda, K) = ?$ 3

②



λ_0 has $lk(\lambda_0, K) = 0$

λ_0 is called

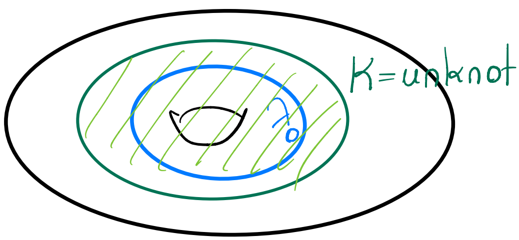
the 0-framed longitude

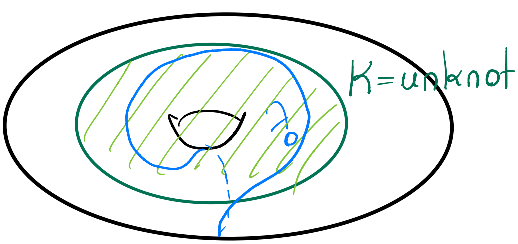
OR Seifert longitude.

Claim: λ_0 is the only (up to isotopy) simple closed curve which has

$$\#(\lambda_0, \mu) = +1$$

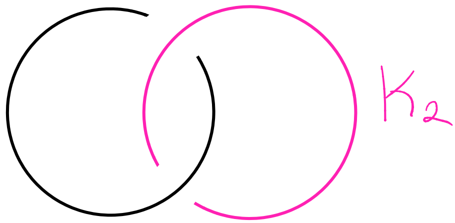
and can be pushed to lie in a Seifert Surface for K.



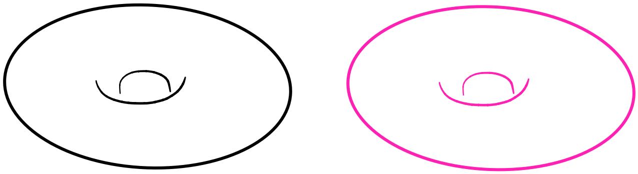


Any other choice intersects all Seifert surfaces of K.

Surgery on links in S^3



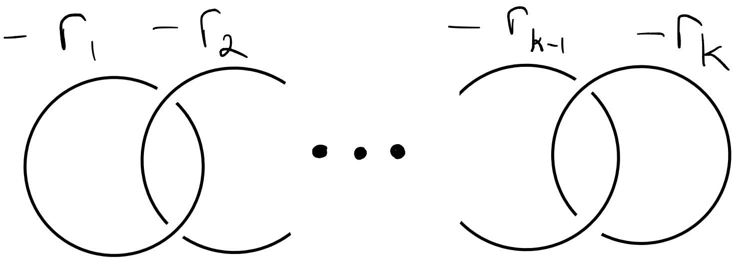
$L = K_1 \cup K_2$



$$S^3 - \nu(L) = S^3 - \left[\nu(K_1) \sqcup \nu(K_2)\right]$$

Given an n-component link $L \hookrightarrow S^3$, we can perform Dehn surgery on L to obtain $L(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$ where a torus T_i is glued to $\partial(\nu(K_i))$ along the slope p_i/q_i

Ex: Lens Spaces:

$X^4 =$ 

Continued Fraction:

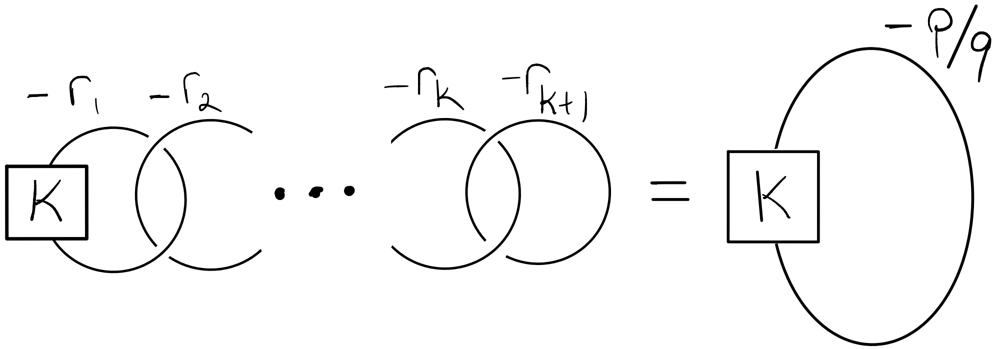
$$[r_1, r_2, \ldots, r_k] = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}} = \frac{p}{q}$$

$$r_i \geq 2$$

$$\Longrightarrow \partial X = L(p,q)$$

Fact: $L(p,q) = S^3_{-p/q}(U)$

Recall: Corollary: Let $(p, q) = 1$

$\Longrightarrow$ 

where $\dfrac{p}{q} = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}}$

Proof: Just slam-dunk $(k-1)$-times

Ex:

$$L(3,2) = S^3_{-5/2}(U)$$

$$3 - \frac{1}{2} = \frac{5}{2}$$

We can be think the surgery as 2 surgeries on disjoint unknots.

Thm : Corollary + [Likorish – Wallace]

Any closed, oriented 3-man.
can be constructed via
a Dehn surgery on a link in S^3.
with all surgery coeff-s being integers.

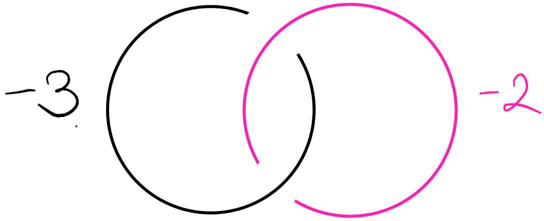
Fact:

M = Dehn surgery on L
M' = Dehn surgery on L'

$\Rightarrow M \# M' =$

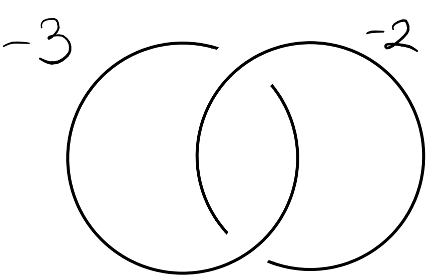
Dehn surgery on $L \cup L'$

Ex:

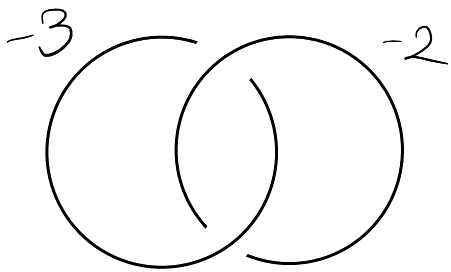


Q: What does this have to do with our 4-man. pictures from before?

From Handle diagrams for 4-man. to Surgery Diagram for 3-man.



X^4



M^3

Link diagrams with each component labeled with an integer have both meanings

4-dimensionally (handle diagram) and

3-dimensionally (surgery diagram)

Q: How are they related?

A: $\partial X = M$ (as we'll see)

$$\partial B^4 = S^3$$

$$\begin{aligned}\partial B^4 &= \partial(B^2 \times B^2) \\ &= (\partial B^2 \times B^2) \cup (B^2 \cup \partial B^2) \\ &= (S^1 \times B^2) \cup (B^2 \cup S^1)\end{aligned}$$

$$\Longrightarrow S^3 = (S^1 \times B^2) \underset{\varphi}{\cup} (B^2 \cup S^1)$$

Note: With care, we can figure out what φ is..

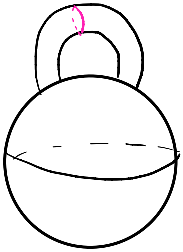
Claim:

Let X be a man. with decomposition $= A \cup B$

where A and B are both manifolds

$$\Rightarrow \partial X \cong \left[\partial(A) - \text{gluing region}\right] \cup \left[\partial(B) - \text{gluing region}\right]$$

Ex:



M^3

3-dim.
handle decomposition

$$\partial M = [\partial(\underbrace{0\text{-}h}_{D^0 \times D^3}) - \text{overlap}] \cup [\partial(\underbrace{1\text{-}h}_{D^1 \times D^2}) - \text{overlap}]$$

$$= \left[\left(\underbrace{(\underbrace{\partial D^0}_{\phi} \times D^3)}_{\phi} \cup \underbrace{(D^0 \times \partial D^3)}_{\{pt\} \times S^2} \right) - \text{attaching region} \right]$$

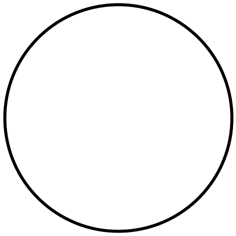
$$\cup \left[(\underbrace{\partial D^1}_{S^0} \times D^2) \cup (D^1 \times \underbrace{\partial D^2}_{S^1}) - \text{attaching region} \right]$$

$$= \left[\underbrace{(S^1 \times I) \cup (D^2 \times S^0)}_{S^2} - \text{attaching region} \right] \quad \text{2 disks}$$

$$\cup \left[(\{2\ pt.\} \times D^2) \cup (D^1 \times S^1) - \text{attaching region} \right] \quad \text{2 disks}$$

$$= [S^2 - \text{attaching region}] \cup [D^1 \times S^1]$$

$$= \text{cylinder} \underset{\varphi}{\cup} \text{cylinder} = \text{torus}$$

Ex: $X =$  ^{n}

$$\partial(X) = [\partial(0\text{-}h) - \text{overlap}] \cup [\partial(2\text{-}h) - \text{overlap}]$$

$$= \Big[\underbrace{(\underbrace{\partial(D^0)}_{\emptyset} \times D^4)}_{\emptyset} \cup (\underbrace{D^0}_{=\{pt\}} \times \underbrace{\partial(D^4)}_{S^3}) - \text{attaching region}\Big]$$

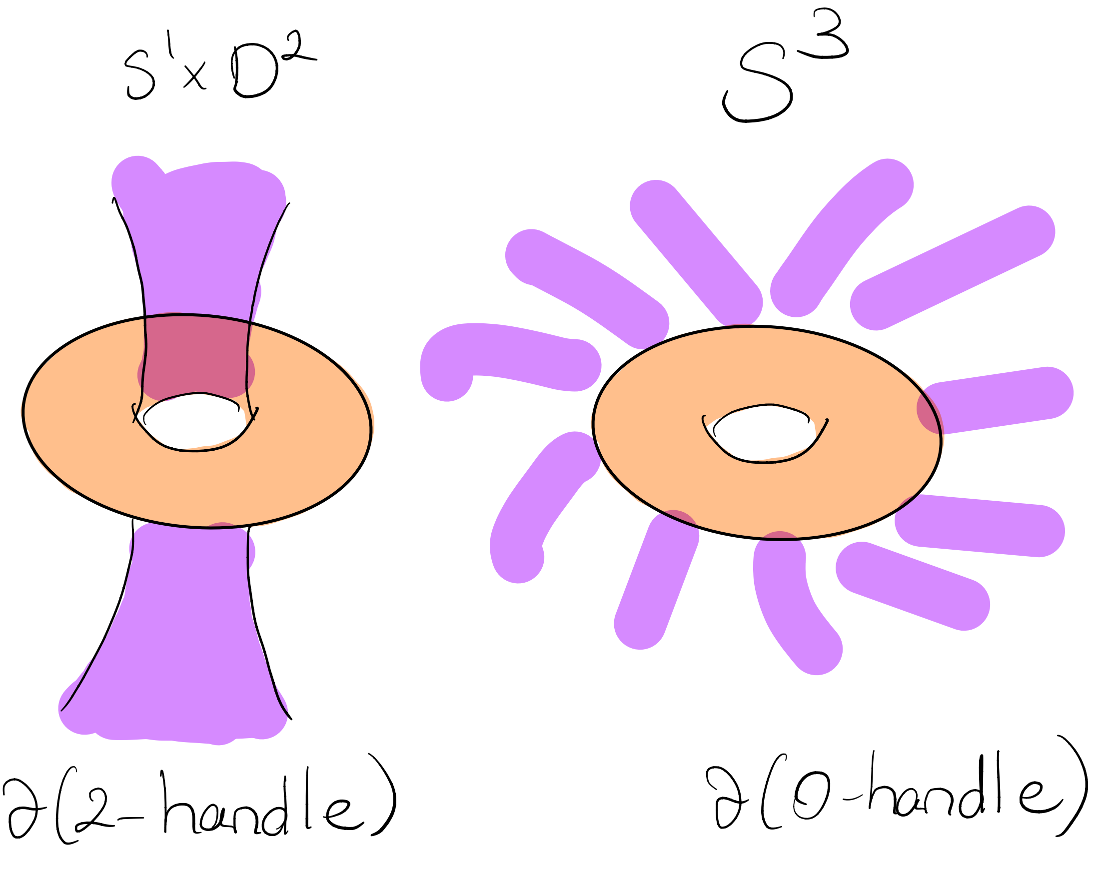
$$\cup \Big[(\partial(D^2) \times D^2) \cup (D^2 \times \partial(D^2)) - \text{attaching region}\Big]$$

$$= [S^3 - \text{attaching region}] \cup [(S^1 \times D^2) \cup (D^2 \times S^1) - \text{attaching region}]$$

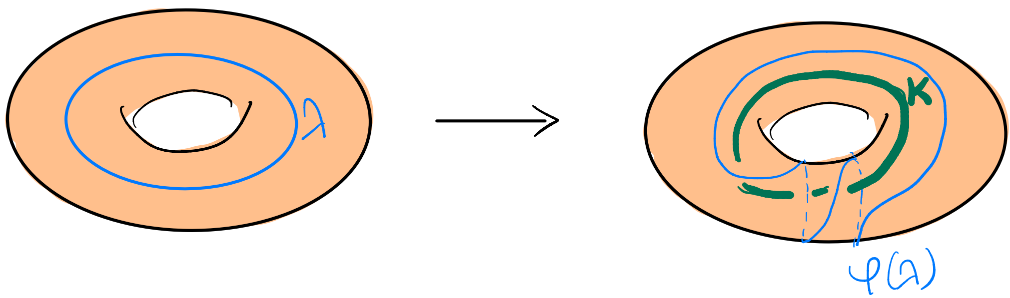
$$= (S^3 - \nu K) \cup_{\varphi} (D^2 \times S^1)$$

So ∂X is certainly a Dehn Surgery.

But what is the framing?



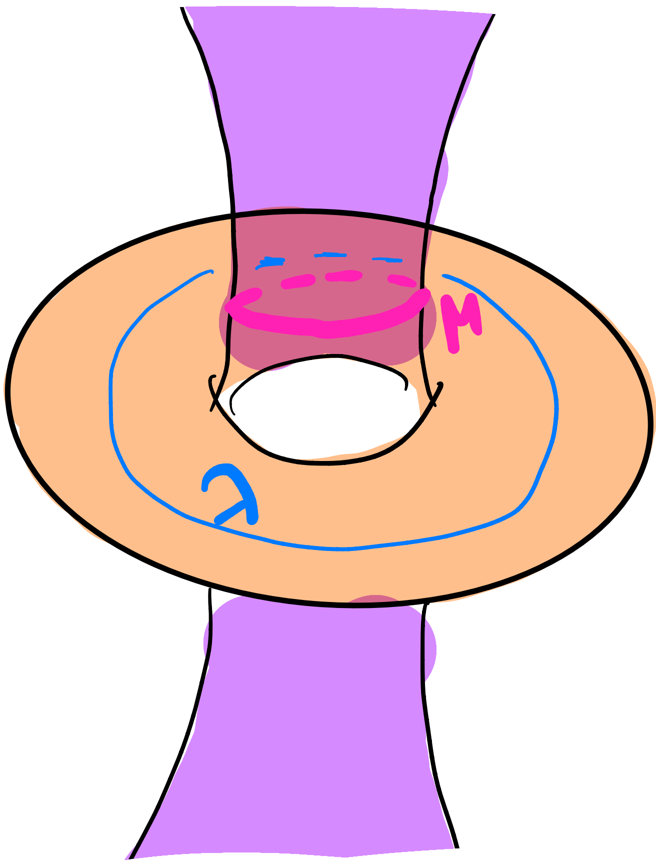
The two orange regions will be identified. How?



φ is determined by where λ goes, which is what is encoded by the framing.

"n" framing $\Rightarrow$ $lk(\varphi(\lambda), K) = n$

We keep track of the identification, but note that the orenge regions are not part of the boundary.



To determine our surgery coeff.,
we need to understand
where the meridian of the
complementary purple torus goes
under φ.

But note that the meridian of
the purple $S^1 \times D^2$ is isotopic to λ.

Thus, $lk(\mu, K) = n$ also.

This is equiv. to performing
n - surgery.

So, we prove that

Proposition:

$$\partial\left(\begin{array}{l}\text{2-h}\\ \text{with framing ``}n\text{''}\\ \text{along } K\end{array} \cup D^4\right) \cong S^3_n(K)$$

Similarly

$$\partial\left(\bigcup_{i=1}^{m}\left(\begin{array}{l}\text{2-h}\\ \text{with framing ``}n_i\text{''}\\ \text{along } K_i\end{array}\right) \cup B^4\right)$$

$$\cong S^3_{n_1,n_2,\dots,n_k}(K_1,K_2,\dots,K_m)$$

Thm:

L, L': $\mathbb{Q}$-framed links
which determine 3-man. as surgery.

If $S^3(L) \cong S^3(L')$
orient. pres.

$\Longrightarrow$ L and L' can be related by

- Rolfsen twist (and undoing them)
- Adding/Removing components framed by ∞
- Isotopy

Note: Nice thm.

In principle, can recover all surgery diagrams producing a given 3-man but not very useful in practice.