Diagonalization

Let Q be a nondegenerate symmetric bilinear form

Recall, if $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the standard basis and $\mathcal{B} = \{b_1, \ldots, b_n\}$ is another basis of $\mathbb{Z}^n$,

then $Q_\mathcal{B} = P^T Q_\mathcal{E} P$ where $P = [b_1 \cdots b_n]$

Moreover, given $v = \sum_{i=1}^n v_i e_i \in \mathbb{Z}^n$, we can express v in the $\mathcal{B}$ basis: $v = \sum_{i=1}^n v_i' b_i$. That is, $[v]_\mathcal{E} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $[v]_\mathcal{B} = \begin{bmatrix} v_1' \\ \vdots \\ v_n' \end{bmatrix}$

Now $P[v]_\mathcal{B} = [b_1 \cdots b_n] \begin{bmatrix} v_1' \\ \vdots \\ v_n' \end{bmatrix} = \sum_{i=1}^n v_i' b_i = \sum_{i=1}^n v_i e_i = [v]_\mathcal{E}$

Hence the map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ transforms $[v]_\mathcal{B}$ into $[v]_\mathcal{E}$
$\varphi(v) = Pv$

So we can view changing basis from $\mathcal{B}$ to $\mathcal{E}$ as the map φ

Claim: φ gives rise to a lattice isomorphism
$\varphi : (\mathbb{Z}^n, Q_\mathcal{B}) \to (\mathbb{Z}^n, Q_\mathcal{E})$ (i.e. a surjective lattice embedding)

proof:

- φ is linear since it is multiplication by P
- $\varphi(u)^T Q_\mathcal{E} \varphi(v) = (Pu)^T Q_\mathcal{E} (Pv) = u^T (P^T Q_\mathcal{E} P) v = u^T Q_\mathcal{B} v$
  $\Rightarrow \varphi$ is a lattice embedding.
- φ is surjective since $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis and $b_i = Pe_i \ \forall i$. $\blacksquare$

So

Changing basis $\mathcal{E} \to \mathcal{B} \iff$ Lattice isomorphism $(\mathbb{Z}^n, Q_\mathcal{B}) \to (\mathbb{Z}^n, Q_\mathcal{E})$

Def: A symmetric bilinear form Q is diagonalizable over $\mathbb{Z}$ if $\exists$ a basis $\mathcal{B}$ of $\mathbb{Z}^n$ such that $Q_\mathcal{B}$ is diagonal (i.e. $\exists$ matrix P s.t. $P^T Q_\mathcal{E} P$ is diagonal)

Ex: Let Q be given by $Q_\mathcal{E} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Then $P^T Q_\mathcal{E} P = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonal

$\Rightarrow Q$ is diagonalizable.

From our discussion so far, we can deduce the following:

Fact: Let Q be diagonalizable with $|\det Q| = 1$

- Q is positive definite $\Leftrightarrow \exists$ basis $\mathcal{B}$ such that $Q_\mathcal{B} = I$
- Q is negative definite $\Leftrightarrow \exists$ basis $\mathcal{B}$ such that $Q_\mathcal{B} = -I$

So by the key observation:

- A positive definite lattice $(\mathbb{Z}^n, Q)$ with $\det Q = 1$ is diagonalizable iff $\exists$ lattice isomorphism $\varphi: (\mathbb{Z}^n, Q) \longrightarrow (\mathbb{Z}^n, I)$
- A negative definite lattice $(\mathbb{Z}^n, Q)$ with $|\det Q| = 1$ is diagonalizable iff $\exists$ lattice isomorphism $\varphi: (\mathbb{Z}^n, Q) \longrightarrow (\mathbb{Z}^n, -I)$

<u>Ex</u>: $Q = \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}$

Note: • $\det Q = 1$

• eigenvalues both negative $\Rightarrow$ negative definite

We can show Q is diagonalizable by finding a lattice isomorphism $\varphi: (\mathbb{Z}^2, Q) \longrightarrow (\mathbb{Z}^2, -I)$

As previously, let $\{f_1, f_2\}$ be the standard basis of the domain and let $\{e_1, e_2\}$ be the standard basis of the codomain.

Let $\varphi(f_1) = x_1 e_1 + x_2 e_2$ and $\varphi(f_2) = y_1 e_1 + y_2 e_2$.

- $-2 = Q(f_1, f_1) = -I(\varphi(f_1), \varphi(f_1)) = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_1^2 - x_2^2$

  $\Rightarrow \quad x_1, x_2 \in \{\pm 1\}$

- $-5 = Q(f_2, f_2) = -I(\varphi(f_2), \varphi(f_2)) = -y_1^2 - y_2^2$

  $\Rightarrow \quad (y_1, y_2) \in \{(\pm 1, \pm 2), (\pm 2, \pm 1)\}$

- $3 = Q(f_1, f_2) = -I(\varphi(f_1), \varphi(f_2)) = -x_1 y_1 - x_2 y_2$

Setting $x_1 = x_2 = 1$ and $y_1 = -2$, $y_2 = -1$ satisfies the equations

$\Rightarrow$ we have a lattice embedding

given by $\varphi(f_1) = e_1 + e_2$,
$\varphi(f_2) = -2e_1 - e_2$

$\Rightarrow \varphi$ is given by $P = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

In the basis $\{\varphi(f_1), \varphi(f_2)\}$, we have

$Q_{\mathcal{B}} = P^T Q_{\mathcal{E}} P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is diagonal