Exam 3 Solutions

1. Let $\mathbf{F} = \sin(\pi y)\mathbf{i} + 3xyz\mathbf{j} + z\mathbf{k}$

   (a) Compute the divergence of $\mathbf{F}$ at $(1, 0, -1)$.

   Solution:

   $$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(\sin(\pi y)) + \frac{\partial}{\partial y}(3xyz) + \frac{\partial}{\partial z}(z) = 3xz + 1$$

   $\text{div } \mathbf{F}(1, 0, -1) = -2$.

   (b) Compute the curl of $\mathbf{F}$ at $(1, 0, -1)$

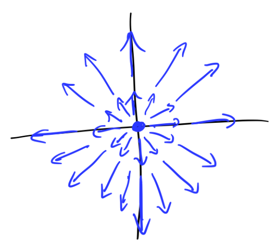
   Solution:

   $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(\pi y) & 3xyz & z \end{vmatrix} = (-3xy, 0, 3yz - \pi\cos(\pi y))$$

   $\text{curl } \mathbf{F}(1, 0, -1) = (0, 0, -\pi)$

   (c) Sketch a vector field in $\mathbb{R}^2$ whose divergence at the origin is positive.

   Solution:



2. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ and C is the portion of the graph of $y = 2x$ from $(0, 0)$ to $(1, 2)$.

   Solution: We can parametrize C by $\mathbf{c}(t) = (t, 2t)$, $0 \le t \le 1$. Thus $\mathbf{c}'(t) = (1, 2)$. Thus

   $$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 (t^2, -2t^2) \cdot (1, 2)\, dt = \int_0^1 -3t^2\, dt = -1$$

3. Let $\mathbf{F} = (\sin(xy) + xy\cos(xy))\mathbf{i} + (x^2\cos(xy))\mathbf{j}$.

   (a) $\mathbf{F}$ is conservative. Explain why.

Solution: Let $P(x, y) = \sin(xy) + xy\cos(xy)$ and $Q(x, y) = x^2\cos(xy)$. Then $\dfrac{\partial P}{\partial y} = 2x\cos(xy) - x^2 y\sin(xy)$ and $\dfrac{\partial Q}{\partial x} = 2x\cos(xy) - x^2 y\sin(xy)$. Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and $\mathbf{F}$ is C^1 on all of $\mathbb{R}^2$, which is simply connected, $\mathbf{F}$ is conservative.

(b) Compute the line integral of $\mathbf{F}$ along the closed curve C given by $x^2 + y^2 = 1$ with the counterclockwise orientation. (Hint: Use part (a))

Solution: Since $\mathbf{F}$ is conservative and C is closed, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} = 0$, by the fundamental theorem of line integrals.

4. Let S be the surface parametrized by $\mathbf{r}(u, v) = (u, v, u^2 + v^2)$, where $u^2 + v^2 \leq 1$.

(a) Calculate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$.

Solution: $\mathbf{r}_u = (1, 0, 2u)$ and $\mathbf{r}_v = (0, 1, 2v)$. Thus $\mathbf{r}_u \times \mathbf{r}_v = (-2u, -2v, 1)$. Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(u, v, u^2 + v^2) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

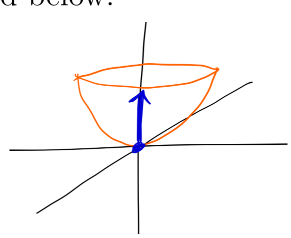
$$= \iint_D (-u, -v, -u^2 - v^2) \cdot (-2u, -2v, 1)\, du\, dv = \iint_D u^2 + v^2\, du\, dv$$

Switching to polar coordinates, we have

$$\iint_D u^2 + v^2\, du\, dv = \int_0^{2\pi} \int_0^1 (r^2) r\, dr\, d\theta = \frac{\pi}{2}$$

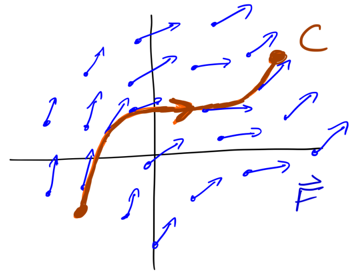
(b) Notice that S is a portion of the graph of $z = x^2 + y^2$, which has a bowl shape. Does the orientation of S point into the bowl or out of the bowl? Explain your reasoning.

Solution: We have to determine which direction the normal vectors $\mathbf{r}_u \times \mathbf{v} = (-2u, -2v, 1)$ point. When $u = 0$ and $v = 0$, $\mathbf{r}_u \times \mathbf{v}(0, 0) = (0, 0, 1)$. This occurs at $\mathbf{r}(0, 0) = (0, 0, 0)$ and is depicted below.



Since this vector points into the bowl, the orientation points into the bowl.

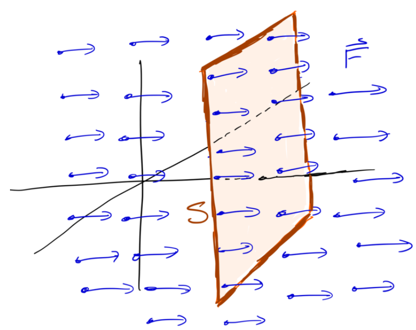
5. (a) Let $C \subset \mathbb{R}^2$ be a flow line of the vector field $\mathbf{F}$ depicted below.



Is the work done by $\mathbf{F}$ in moving a particle along C positive, negative, or zero? Explain your reasoning.

Solution: The work done is positive, since the flow of $\mathbf{F}$ is moving in the same direction as the orientation of C.

(b) Consider the surface S and vector field $\mathbf{F}$ in $\mathbb{R}^3$ depicted below.



Suppose S has the leftward-pointing orientation. Is the flux of $\mathbf{F}$ through S positive, negative or zero? Explain your reasoning.

Solution: The flux is negative, since the flow of $\mathbf{F}$ is moving against the orientation of S.