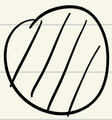
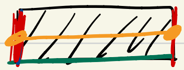
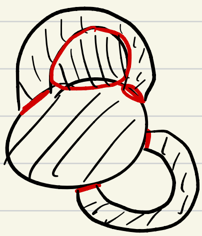
Handlebody decompositions

Theorem: Every manifold admits a handle decomposition

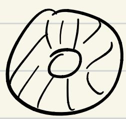
Surfaces

0-handle	1-handle	2-handle
		
$D^0 \times D^2$	$D^1 \times D^1$	$D^2 \times D^0$
	$\partial D^1 \times D^1$ = Attaching Region	$\partial D^2 \times D^0$ = Attaching Region
	$\partial D^1 \times \{0\}$ = core	

Ex:



$\approx$



Annulus

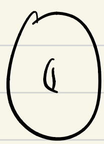
$S^1 \times I$

Ex:

$S^2 = 0h \cup 2h$

Ex:

$0h \cup 1h \cup 1h \cup 2h \rightsquigarrow T^2$



Ex:

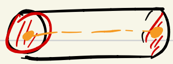
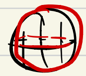
$\approx$

Mobius band

Fact: $\chi(S) = \#0\text{-handles} - \#1\text{-handles} + \#2\text{-handles}$

(Disks) (bands)

3-manifolds

0-handle	1-handle	2-handle	3-handle
			
$D^3 \times D^0$	$D^1 \times D^2$	$D^2 \times D^1$	
	$\partial D^1 \times D^2 =$ Att. Region	$\partial D^2 \times D^1 =$ Att. Region	
	$\partial D^1 \times \{0\} =$ Core	$\partial D^2 \times \{0\} =$ Core	

Ex:

$S^3 =$ 0-handle $\cup$ 3-handle

4-manifolds

0-handle	1-handle	2-handle	3-handle	4-handle
D^4	$D^1 \times D^3$	$D^2 \times D^2$	$D^3 \times D^1$	$D^4 \times D^0$.

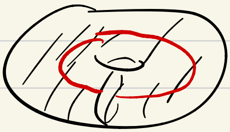
1-handles must be attached to $\partial D^4 = S^3 = \mathbb{R}^3 \cup \{\infty\}$

The attaching region is $\partial D^1 \times D^3 = S^0 \times D^3$

  Think "wormhole"

The attaching region of a 2-handle is $\partial D^2 \times D^2 = S^1 \times D^2$

Attaching circle is $S^1 \times \{0\}$



Data needed for a 2-handle attachment is:

- Attaching circle (can be knots)
- Framing: How do we glue the attaching region $S^1 \times D^2$ in 2h to a nbhd of the knot in $\partial B^4 = S^3$.
  There are $\mathbb{Z}$-many ways to do this

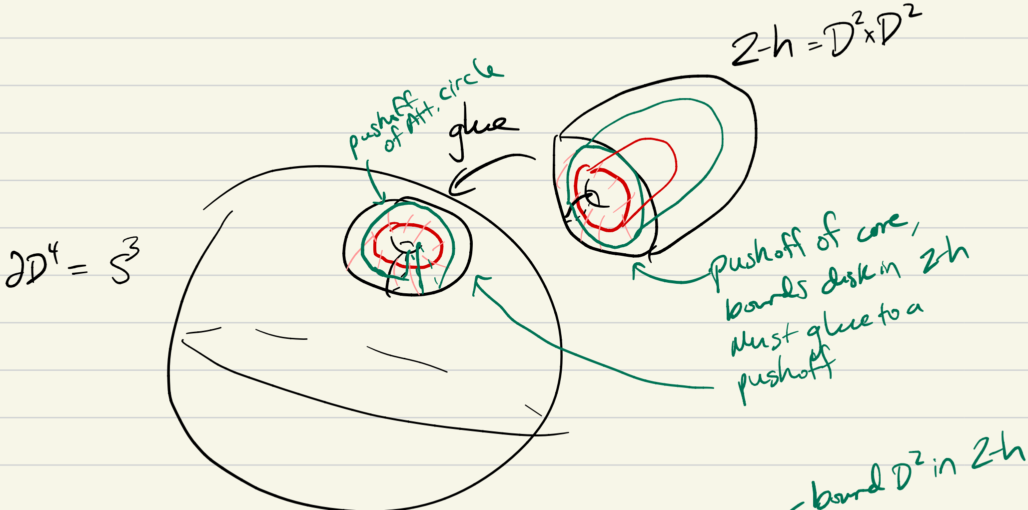
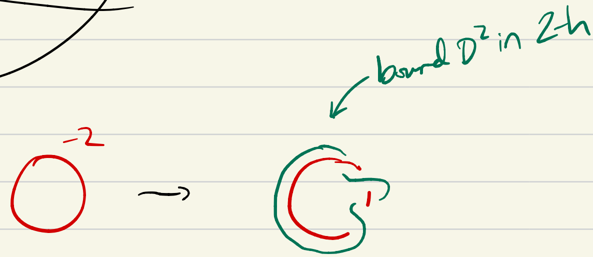
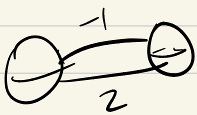


Diagram in S^3:

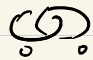


Ex: Some blueprints for 4-manifolds:



Ex: Well-known 4-manifolds: $S^4 = $ 0-h $\cup$ 4-h

$S^1 \times S^3 = $ 0-h $\cup$ 1-h $\cup$ 3-h $\cup$ 4-h

$S^2 \times S^2 = $  $\cup$ 4h

$\mathbb{CP}^2 = \bigcirc^{1}$

$\overline{\mathbb{CP}^2} = \bigcirc^{-1}$