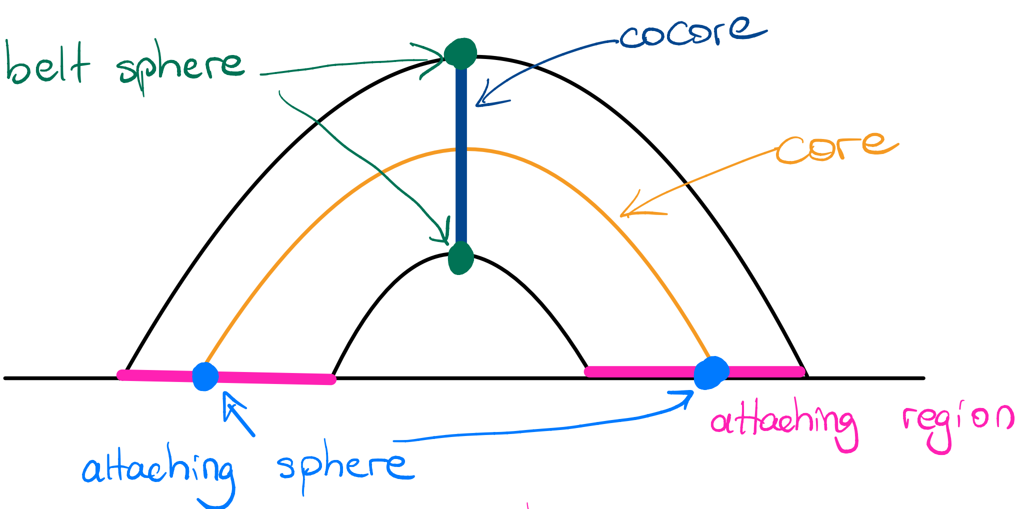
Handlebody Decomposition

Handle Decomposition

Thm: Every manifold admits a handle decomposition.

n-dim. k-handle: $h_k^n := D^k \times D^{n-k}$

Attach $(k+1)$-handles to k-handles.



attaching region: $\partial D^k \times D^{n-k}$

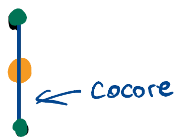
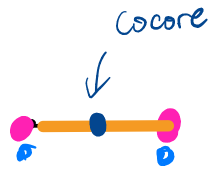
core : $D^k \times \{0\}$

attaching sphere: $\partial(\text{core}) = \partial D^k \times \{0\} = S^{k-1} \times \{0\}$

cocore : $\{0\} \times D^{n-k}$

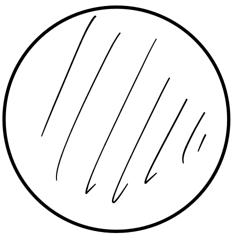
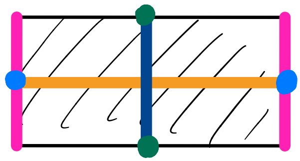
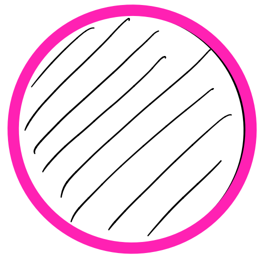
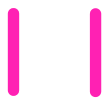
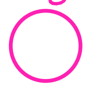
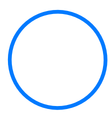
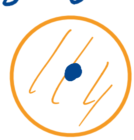
belt sphere : $\{0\} \times \partial(D^{n-k}) = \{0\} \times S^{n-k-1}$

1- manifolds

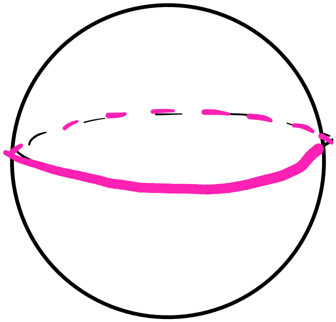
	0-handle	1-handle
	$D^0 \times D^1$ $D^1 = [0,1]$ 	$D^1 \times D^0$ 
attaching region	$\partial D^0 \times D^1 = \emptyset$	$\partial D^1 \times D^0$ $= S^0 \times D^0$
core	$D^0 \times \{0\}$	$D^1 \times \{0\} = D^1$ 
attaching sphere ∂(core)	$\partial(D^0 \times \{0\}) = \emptyset$	$\partial(D^1 \times \{0\}) = S^0$
cocore	$\{0\} \times D^1 = D^1$	$\{0\} \times D^0 = \{pt\}$
belt sphere ∂(cocore)	$\partial(\{0\} \times D^1)$ $= \{0\} \times \partial D^1 = S^0$	$\partial(\{0\} \times D^0)$ $= \{0\} \times \partial D^0 = \emptyset$

2-manifolds: Surfaces

We'll only need 2-dimensional handles

	0-handle	1-handle	2-handle
	$D^0 \times D^2$ 	$D^1 \times D^1$ $D^1 = I =$  	$D^2 \times D^0$ 
attaching region		$\partial D^1 \times D^1$ $= S^0 \times D^1$ 	$\partial D^2 \times D^0$ $= S^1 \times D^0$ S^1 
core	$D^0 \times \{0\}$	$D^1 \times \{0\}$	$D^2 \times \{0\}$ 
attaching sphere ∂(core)	$\partial(D^0 \times \{0\}) = \emptyset$	$\partial(D^1 \times \{0\}) = S^0$ 	$\partial(D^2 \times \{0\}) = S^1$ 
cocore	$\{0\} \times D^2$	$\{0\} \times D^1 = D^1$	$\{0\} \times D^0 = \{pt\}$ 
belt sphere ∂(cocore)	$\partial(\{0\} \times D^2)$ $= \{0\} \times \partial D^2 = S^1$	$\partial(\{0\} \times D^1)$ $= \{0\} \times \partial D^1 = S^0$	$\partial(\{0\} \times D^0)$ $= \{0\} \times \partial D^0 = \emptyset$

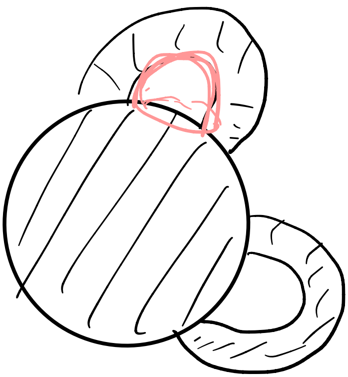
Ex: S^2



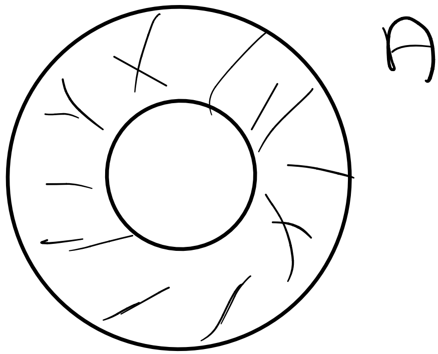
$$S^2 = h_0 \cup h_2$$

$$\chi(S^2) = 1 - 0 + 1 = 2$$

Ex: Annulus $= S^1 \times I$

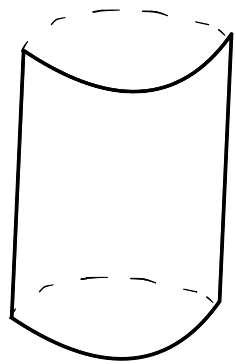


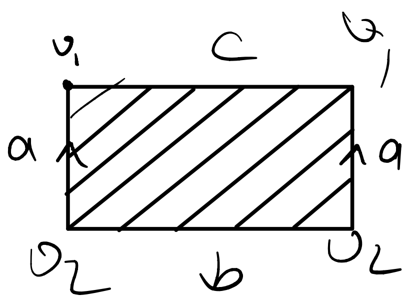
$\approx$



Annulus $= S^1 \times I$

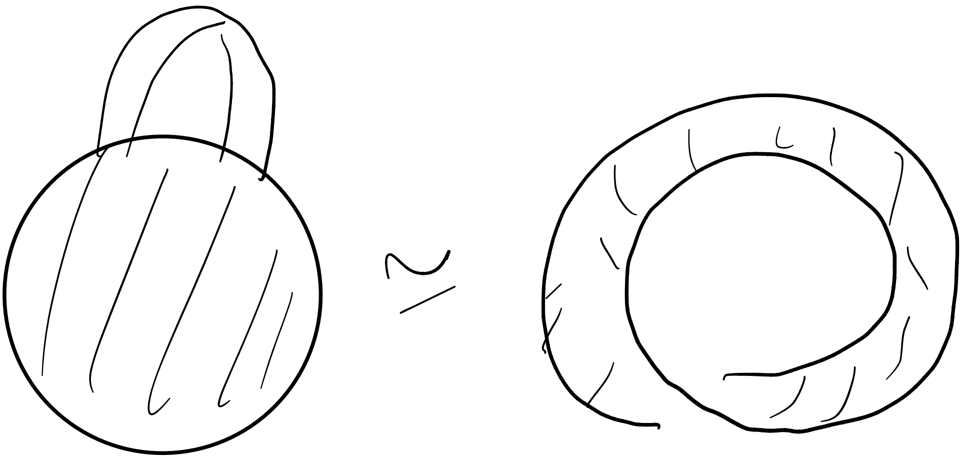
$\chi(A) = 1 - 2 + 1 = 0$





$2 - 3 + 1 = 0$

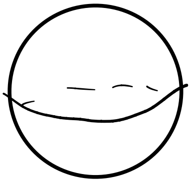
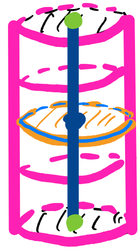
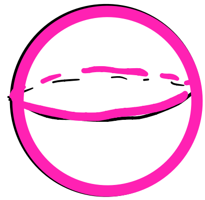
Ex: Möbius Band



$$\chi(MB) = 1 - 1 - 0 = 0$$

$$\boxed{\chi(S) = \underbrace{\#\,0\text{-handles}}_{\text{disks}} - \underbrace{\#\,1\text{-handles}}_{\text{bands}} + \#\,2\text{-handles}}$$

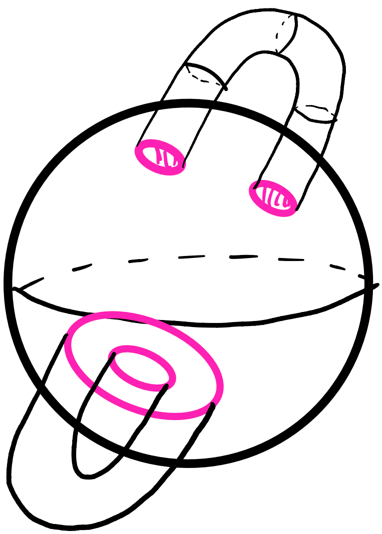
3-manifolds

	0-handle	1-handle	2-handle	3-handle
	$D^0 \times D^3$ 	$D^1 \times D^2$ 	$D^2 \times D^1$ 	$D^3 \times D^0$ 
attaching region		$\partial D^1 \times D^2$ $= S^0 \times D^2$	$\partial D^2 \times D^1$ $= S^1 \times D^1$ 	$\partial D^3 \times D^0$ $= S^2 \times D^0$ 
core	$D^0 \times \{0\}$	$D^1 \times \{0\}$ 	$D^2 \times \{0\}$ 	$D^3 \times \{0\}$
attaching sphere	$\partial(D^0 \times \{0\}) = \emptyset$	$\partial(D^1 \times \{0\}) = S^0$ $= \partial D^1 \times \{0\}$ 	$\partial(D^2 \times \{0\}) = S^1$ 	$\partial(D^3 \times \{0\}) = S^2$ 
cocore	$\{0\} \times D^3$	$\{0\} \times D^2$	$\{0\} \times D^1$	$\{0\} \times D^0$
belt sphere ∂(cocore)	$\partial(\{0\} \times D^3)$ $= \{0\} \times \partial D^3$ $= S^2$	$\partial(\{0\} \times D^2)$ $= \{0\} \times \partial D^2$ $= S^1$	$\partial(\{0\} \times D^1)$ $= \{0\} \times \partial D^1$ $= S^0$	$\partial(\{0\} \times D^0)$ $= \{0\} \times \partial D^0$ $= \emptyset$

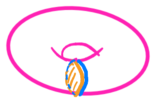
Ex: S^3

$$S^3 = 0\text{-h} \cup 3\text{-h}$$

Ex:



4-manifolds:

	0-handle	1-handle	2-handle	3-handle	4-handle
	$D^0 \times D^4 = D^4$	$D^1 \times D^3$	$D^2 \times D^2$	$D^3 \times D^1$	$D^4 \times D^0 = D^4$
attaching region	$\partial D^0 \times D^4 = \emptyset$	$\partial D^1 \times D^3 = S^0 \times D^3$  D^3 D^3	$\partial D^2 \times D^2 = S^1 \times D^2$  solid torus	$\partial D^3 \times D^1 = S^2 \times D^1$	$\partial D^4 \times D^0 = S^3 \times D^0 = S^3$
core	$D^0 \times \{0\}$	$D^1 \times \{0\}$	$D^2 \times \{0\}$ 	$D^3 \times \{0\}$	$D^4 \times \{0\}$
attaching sphere	$\partial(D^0 \times \{0\}) = \emptyset$	$\partial(D^1 \times \{0\}) = S^0$	$\partial(D^2 \times \{0\}) = S^1$	$\partial(D^3 \times \{0\}) = S^2$	$\partial(D^4 \times \{0\}) = S^3$
cocore	$\{0\} \times D^4 = D^4$	$\{0\} \times D^3 = D^3$	$\{0\} \times D^2 = D^2$	$\{0\} \times D^1 = D^1$	$\{0\} \times D^0 = D^0$
belt sphere ∂(cocore)	$\partial(\{0\} \times D^4) = \{0\} \times \partial D^4 = S^3$	$\partial(\{0\} \times D^3) = \{0\} \times \partial D^3 = S^2$	$\partial(\{0\} \times D^2) = \{0\} \times \partial D^2 = S^1$	$\partial(\{0\} \times D^1) = \{0\} \times \partial D^1 = S^0$	$\partial(\{0\} \times D^0) = \{0\} \times \partial D^0 = \emptyset$

Remarks

① 1-handles must be attached to

$$\partial(0\text{-handle}) = \partial(D^0 \times D^4)$$
$$= (\partial D^0 \times D^4) \cup (D^0 \times \partial D^4)$$
$$= \emptyset \cup (D^0 \times S^3) = D^0 \times S^3$$
$$= S^3 = \mathbb{R}^3 \cup \{\infty\}$$

The attaching region of the 1-handle:

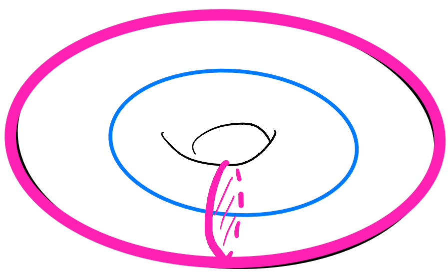
$$\partial D^1 \times D^3 = S^0 \times D^3$$

think "warmhole"

② The attaching region of 2-handle:

$\partial D^2 \times D^2 = S^1 \times D^2$

Attaching circle: $S^1 \times \{0\}$



Data needed for a 2-handle attachment:

① Attaching circle (can be knots)

② Framing: How do we glue the attaching region $S^1 \times D^2$ to a nbhd of the knot in $\partial B^4 = S^3$

There are $\mathbb{Z}$-many ways to do this.

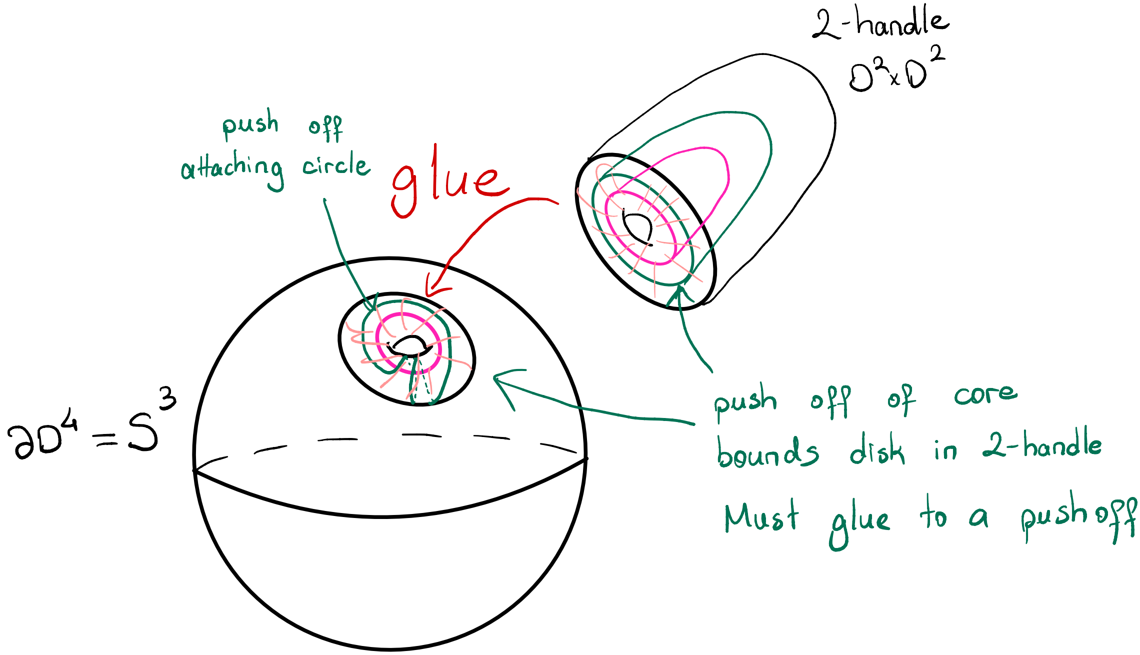
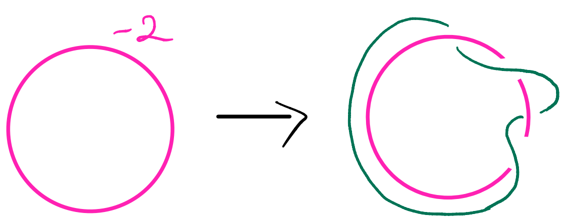
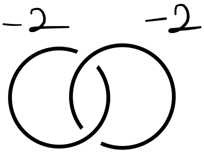


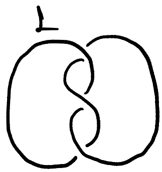
Diagram in S^3

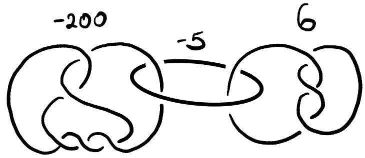


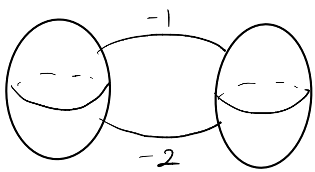
bound D^2
in 2-handle

Ex: Some blue prints for 4-man.s





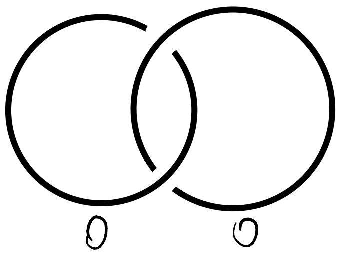




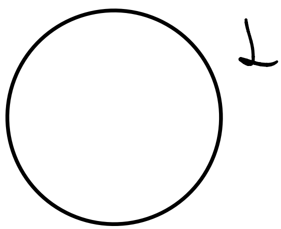
Ex: Some well-known 4-man.s

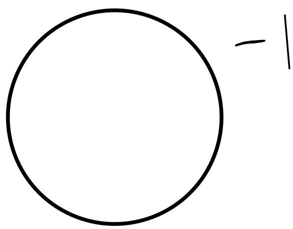
$S^4 = 0\text{-handle} \cup 4\text{-handle}$

$S^1 \times S^2 = 0\text{-h} \cup 1\text{-h} \cup 3\text{-h} \cup 4\text{-h}$

$S^2 \times S^2 =$ 

$\cup\ 4\text{-h}$

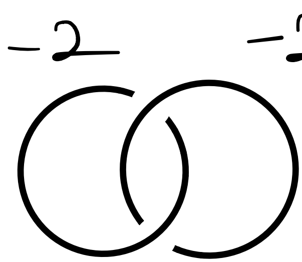
$\mathbb{CP}^2 =$ 

$\overline{\mathbb{CP}^2} =$ 

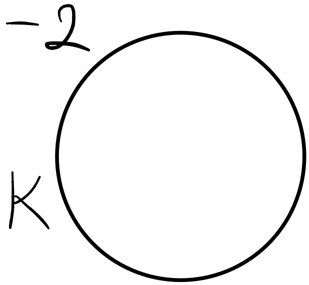
Remark:

All 3-man.s are boundaries of 4-man.s. without 1- OR 3-handles.

Let X be a 4-man. with bdry which is build from 0- and 2-handles.



Then this diagram can also be viewed as a surgery diagram for ∂X



means

remove $\nu(K)$ (tubular nbhd) from S^3

replace it with $S^1 \times D^2$

in a certain way

(adding some twist on $\underbrace{\partial(S^1 \times D^2)}_{S^1 \times S^1 = T^2}$)

(drilling and filling)

This is called "Dehn Surgery"

(later)