Intro. to Homology

Big-Picture: Homology is an algebraic topological gadget defined for a topological space.

It is an invariant under homeomorphism actually under homotopy.

homotopy: $\exists H : X \times [0,1] \to Y$ s.t.

$$H(x,0) = f(x)$$
$$H(x,1) = g(x)$$

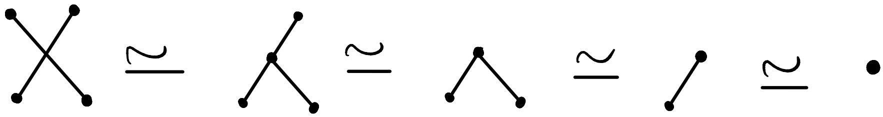
X and Y homotopic (homotopy equivalent)

If $\exists f : X \to Y$ and $\exists g : Y \to X$
s.t. $g \circ f \simeq id_X$

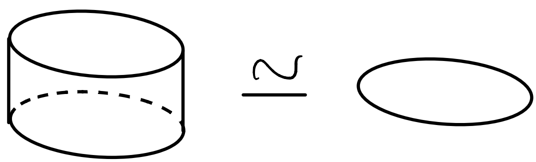
$$f \circ g \simeq id_Y$$

Intuitively, homotopy allows bending, shrinking, expanding (doesn't need to be injective OR surjective)

Examples:

1. 

2. $A \simeq \triangle \simeq \bullet$

3. 

4. $B^n \simeq \bullet$

This will be usefull since to compute homology, we can consider a simpler manifold homotopic to X, which has the same homology but easier to compute.

Cell Decomposition:

These are the building blocks for cellular homology.

cell complex (CW-complex)

A topological space made up of pieces, called skeletons, together with a gluing restriction.

Cells:

0-cell: a point "•"

1-cell: an interval $[0,1]$ 

2-cell: a disk

$\vdots$

n-cell: B^n

Skeletons:

0-skeleton $:= X^0$: Finite union of 0-cells.

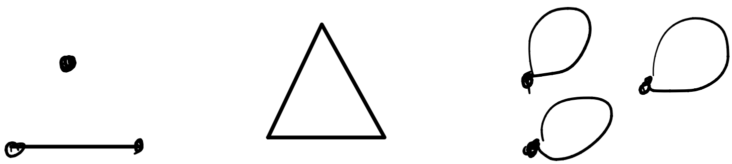
some number of

1-skeleton $:= X^1 := X^0 \cup_{\varphi}$ (1-cells)

$\partial([0,1])$ must be glued to X^0

$$\varphi : \partial([0,1]) \longrightarrow X^0$$

Ex:

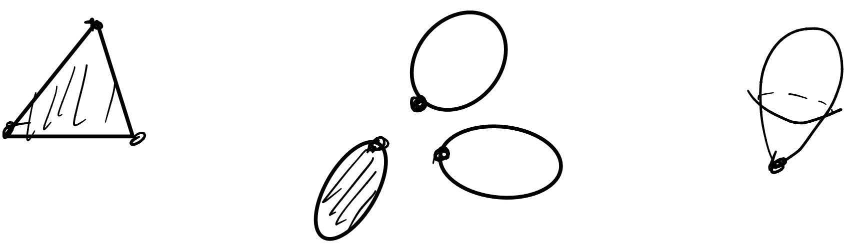


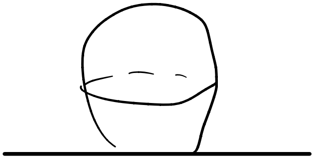
some number of

2-skeleton $:= X^2 := X^1 \cup_{\varphi}$ 2-cells

$$\varphi : \partial(2\text{-cell}) = \partial(D^n) = S^{n-1} \longrightarrow X^1$$

Ex:





unnecessary

badly behaved

n-skeleton $:= X^n := X^{n-1} \bigcup_{\varphi_i}$ some n-cells

$$\varphi : \partial B_i^n \longrightarrow X^{n-1}$$

Remark: $X = X^0 \cup X^1 \cup X^2 \cup \cdots$

can go forever

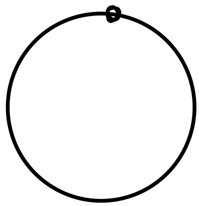
but we'll consider n-dim. man.s.

In that case, we don't add any cells after the n-th stage.

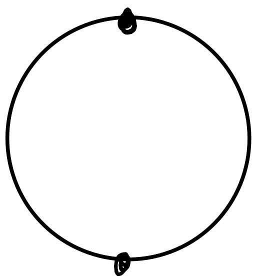
i.e. $X^n = X^{n+1} = X^{n+2} = \cdots$

Examples:

1) a



b



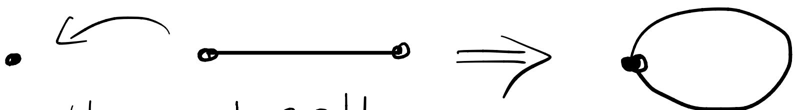
② $S^n = \underbrace{\underbrace{0\text{-cell}}_{X^0} \cup_{\varphi} n\text{-cell}}_{X^n}$

$\varphi : \partial B^n \longrightarrow \{pt\}$

$X^0 = X^1 = X^2 = \cdots = X^{n-1}$

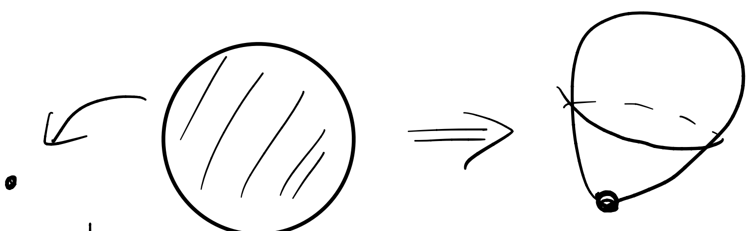
No k-cell for $1 < k < n$

S^1 :



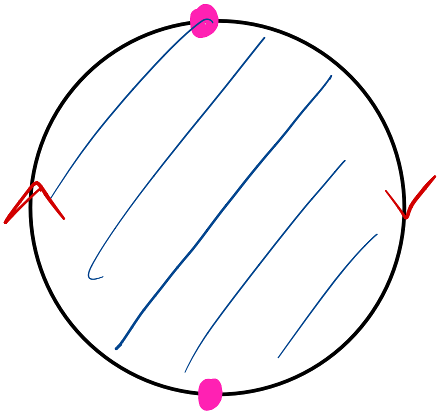
$\underbrace{0\text{-cell}}_{X^0}$ $\quad$ 1-cell $\quad$ S^1

S^2 :



0-cell $\quad$ S^2

③ $\mathbb{RP}^2$



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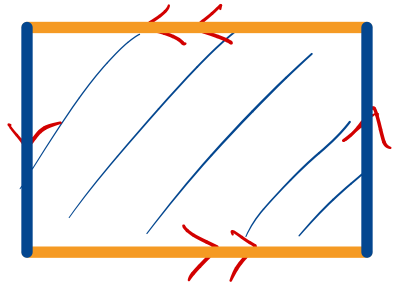
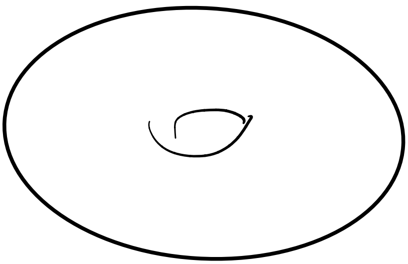
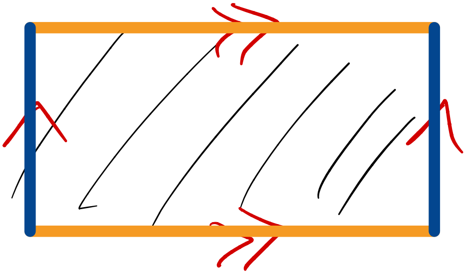


④





Euler Characteristic

$X =$ a (finite) cell-complex

$$\Longrightarrow \boxed{\chi(X) = \sum_{i=0}^{n} (-1)^n \#(i\text{-cells of } X)}$$

where n is the smallest number s.t.

$$X^n = X^{n+1} = X^{n+2} = \cdots$$

If X is a man. $n = \dim(X)$

$$\boxed{\chi(X) = \sum_{i=1}^{n} (-1)^n \operatorname{rank}(H_n(X, \mathbb{Z}))}$$

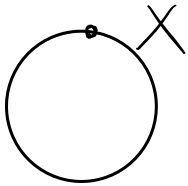
$$H_n(X, \mathbb{Z}) = \mathbb{Z}^k \oplus \text{torsion}$$

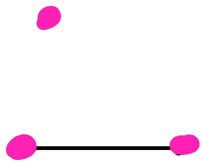
$$\operatorname{rank}(H_n(X, \mathbb{Z})) = \#\ \mathbb{Z} \text{ summands.}$$

Examples:

1

(a)



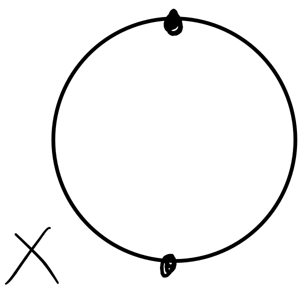


1 0-cell

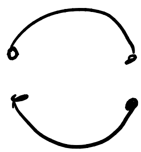
1 1-cell

$$\chi(X) = 1 - 1 = 0$$

(b)



2 0-cells



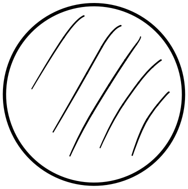
2 1-cells

$$\chi(X) = 2 - 2 = 0$$

2 $\quad S^n$

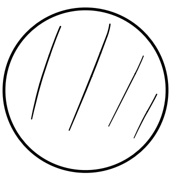
$n = 2$:

• $\qquad$ 1 $\quad$ 0-cell

 $\qquad$ 1 $\quad$ 2-cell

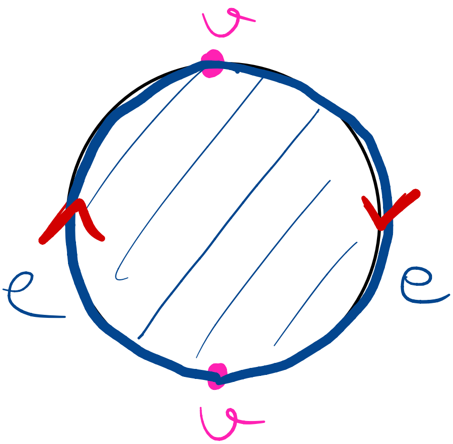
In general for S^n

• $\qquad$ 1 $\quad$ 0 cell

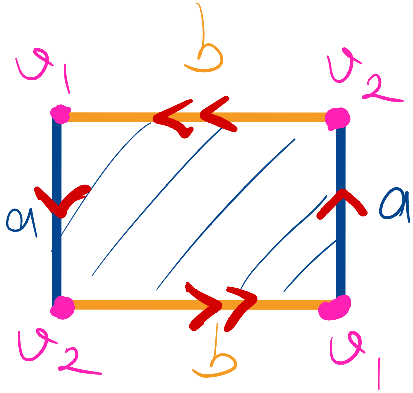
 D^n $\qquad$ 1 $\quad$ n-cell

$$\Rightarrow \chi(S^n) = \begin{cases} 2 & \text{if } \quad n = \text{even} \\ 0 & \text{if } \quad n = \text{odd} \end{cases}$$

③ $\mathbb{RP}^2$



OR

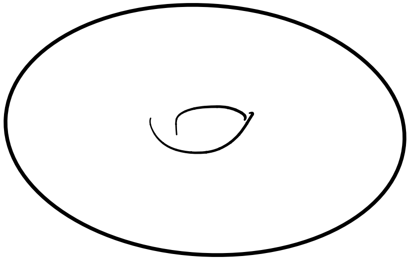


1	0-cell		2	0-cells
1	1-cell	OR	2	1-cells
1	2-cell		1	2-cell

$$\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$$

$$\chi(\mathbb{RP}^2) = 2 - 2 + 1 = 1$$

4



T^2

1 0-cell

2 1-cells

1 2-cell

$$\chi(T^2) = 1 - 2 + 1 = 0$$

Homology is a stronger inv. than Euler characteristic.
It is an alg. gadget which counts # n-dim. holes.

Alg. gadget: the homology of a topological space will be a series of abelian groups $H_n(X)$ one for each $n \in \mathbb{Z}$

Chain Complex

$$\boxed{\cdots \xrightarrow{d_3} C_3 \xrightarrow{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \xrightarrow{d_{-1}} C_{-1} \longrightarrow \cdots}$$

d_i : homomorphisms
C_i : abelian groups s.t.

$$\boxed{\operatorname{Im}(d_i) \subset \operatorname{Ker}(d_{i-1})} \quad (*)$$

$$* \iff d_{i-1} \circ d_i = 0$$

Notation: Often we'll drop the subscripts of d_i's since it is usually clear which d_i we are interested in.

So $d^2 = 0$

Our chain complex will be defined in terms of topology:

$C_i(X)$: cell complexes

$$\boxed{C_i(X) = \text{the (abelian) group of "formal sums" of (oriented) } i\text{-cells in } X.}$$

These sums can be taken with coefficients in $\mathbb{Z}$ (usually), $\mathbb{Q}$ OR $\mathbb{R}$ (or even wilder coefficients)

What is the map d ?

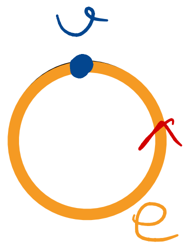
$$C_i \xrightarrow{d_{i-1}} C_{i-1}$$

$$\boxed{d_{i-1}(i\text{-cell}) = \sum \begin{array}{l}\text{oriented} \\ (i-1)\text{-cells} \\ \text{making up} \\ \text{boundary} \\ \text{the } i\text{-cell}\end{array}}$$

Then extend this to $C_i(X)$ to make d_{i-1} a homomorphism.

Remark: This defn. is correct enough for our purposes but really need the notion of degree of maps of sphere to make this rigorous.

Examples:

(1) (a) 

X

$$C_0(X) = \{n \cdot v : n \in \mathbb{Z}\} = \langle v \rangle \cong \mathbb{Z}$$

$$C_1(X) = \{n \cdot e : n \in \mathbb{Z}\} = \langle e \rangle \cong \mathbb{Z}$$

$$C_i(X) = \{n \cdot 0 : n \in \mathbb{Z}\} = 0 \quad \forall i \geq 2$$

$$\cdots \xrightarrow{d_2 = 0} \underset{\substack{\| \\ 0}}{C_2} \xrightarrow{d_1 = 0} \underset{\substack{\| \\ \langle e \rangle}}{C_1} \xrightarrow{d_0} \underset{\substack{\| \\ \langle v \rangle}}{C_0} \xrightarrow{d_{-1} = 0} \underset{\substack{\| \\ 0}}{C_{-1}} \cdots$$

$$e \longmapsto v - v = 0$$

Note: This chain complex is NOT an invariant of X.

Chainging cell decomposition can change $C_n(X)$

$$\cdots \xrightarrow{d_2=0} C_2 \xrightarrow{d_1=0} C_1 \xrightarrow{d_0} C_0 \xrightarrow{d_{-1}=0} C_{-1} \cdots$$

$$C_2 = 0, \quad C_1 = \langle e \rangle, \quad C_0 = \langle v \rangle, \quad C_{-1} = 0$$

$$e \longmapsto v - v = 0$$

$$H_k(X) = 0 \quad \forall\ i \leq -1$$

$$H_0(X) = \ker(d_{-1}) / \operatorname{Im}(d_0) = \langle v \rangle / 0 \cong \mathbb{Z}$$

$$H_1(X) = \ker(d_0) / \operatorname{Im}(d_1) = \langle e \rangle / 0 \cong \mathbb{Z}$$

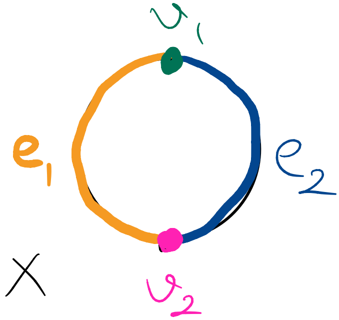
$$H_2(X) = \ker(d_1) / \operatorname{Im}(d_2) = 0/0 = 0$$

$$\vdots$$

$$H_k(X) = 0 \quad \forall\ k \geq 2$$

$$\boxed{H_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}}$$

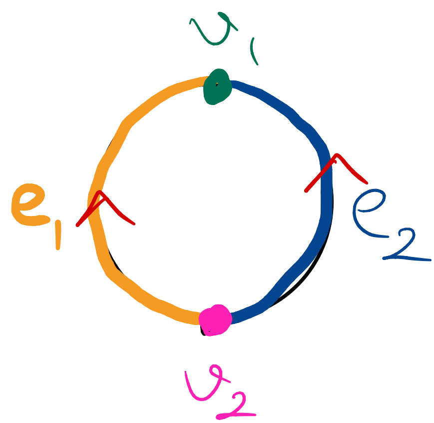
(b)



Orientation ?

Assume our orientation on v_1 and v_2 so that

- If a 1-cell comes out of v_i that corr. to a "+" count
- If a 1-cell is going in v_i that corr. to a "−" count.



$$\cdots \to \underset{\overset{\shortparallel}{0}}{C_3} \xrightarrow{d_2 = 0} \underset{\overset{\shortparallel}{0}}{C_2} \xrightarrow{d_1 = 0} \underset{\overset{\shortparallel}{\langle e_1, e_2 \rangle}}{C_1} \xrightarrow{d_0} \underset{\overset{\shortparallel}{\langle v_1, v_2 \rangle}}{C_0} \xrightarrow{d_{-1} = 0} \underbrace{C_{-1}}_{0} \to \cdots$$

$$\mathrm{Im}(d_0) = \langle v_2 - v_1 \rangle \Longleftarrow \boxed{\begin{array}{l} e_1 \overset{d_0}{\longmapsto} v_2 - v_1 \\ e_2 \overset{d_0}{\longmapsto} v_2 - v_1 \end{array}}$$

This is all we need to extend to all of $C_1 = \langle e_1, e_2 \rangle$

$d_i = 0$ if $i \neq 0$

$\ker(d_0) = \langle e_1 - e_2 \rangle$

since $d_0(e_1 - e_2) = d_0(e_1) - d_0(e_2)$

$= v_2 - v_1 - (v_2 - v_1) = 0$

and all elements in C_1 goint to 0 can be written as $k(e_1 - e_2)$.

$$H_k(X) = 0 \quad \forall\ i \leq -1$$

$$H_0(X) = \ker(d_{-1}) / \operatorname{Im}(d_0) = \langle v_1, v_2 \rangle / \langle v_2 - v_1 \rangle$$

$$= \langle v_1, v_2 - v_1 \rangle / \langle v_2 - v_1 \rangle \cong \langle v_1 \rangle \cong \mathbb{Z}$$

$$H_1(X) = \ker(d_0) / \operatorname{Im}(d_1) = \langle e_1 - e_2 \rangle / 0 \cong \mathbb{Z}$$

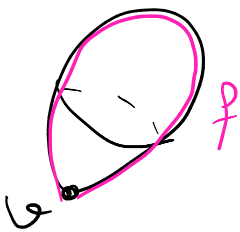
$$H_2(X) = \ker(d_1) / \operatorname{Im}(d_2) \cong 0/0 \cong 0$$

$$\vdots$$

$$H_k(X) = 0 \quad \forall\ k \geq 2$$

$$\boxed{H_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}}$$

2. $S^2 = \underset{v}{0\text{-cell}} \cup \underset{f}{2\text{-cell}}$



$$\cdots \to C_3 \xrightarrow{d_2 = 0} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0 = 0} C_0 \xrightarrow{d_{-1} = 0} C_{-1} \to \cdots$$

$$C_3 = 0, \quad C_2 = \langle f \rangle, \quad C_1 = 0, \quad C_0 = \langle v \rangle, \quad C_{-1} = 0$$

$$f \longmapsto 0$$

$H_k(S^2) = 0 \quad \forall\ i \le -1$

$H_0(S^2) = \ker(d_{-1}) / \mathrm{Im}(d_0) = 0/0 = 0$

$H_1(S^2) = \ker(d_0) / \mathrm{Im}(d_1) = 0/0 = 0$

$$H_2(S^2) = \ker(d_1)/\mathrm{Im}(d_2) = \langle f\rangle/0 \cong \mathbb{Z}$$

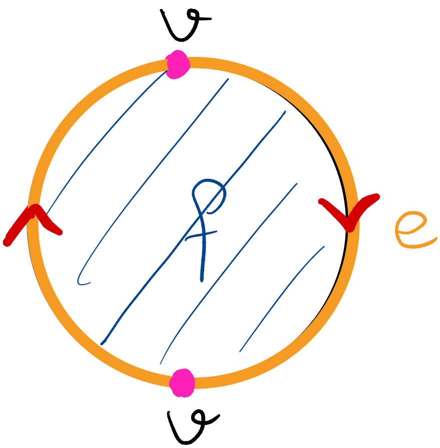
$$H_3(S^2) = \ker(d_2)/\mathrm{Im}(d_3) = 0/0 = 0$$

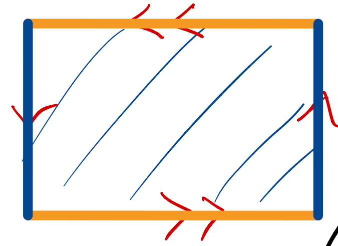
$$\vdots$$

$$H_k(S^2) = 0 \qquad \forall\ k \geq 3$$

$$\Rightarrow H_k(S^2) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

③ $\mathbb{RP}^2$



(OR )

$$\cdots \longrightarrow C_3 \xrightarrow[0]{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \xrightarrow[0]{d_{-1}} C_{-1} \longrightarrow \cdots$$

$$C_3 = 0,\quad C_2 = \langle f \rangle,\quad C_1 = \langle e \rangle,\quad C_0 = \langle v \rangle,\quad C_{-1} = 0$$

$$f \overset{d_1}{\longmapsto} e + e$$

$$e \overset{d_0}{\longmapsto} v - v = 0$$

$H_n(\mathbb{RP}^2) = ?$ HW.

Homology:

Note that the chain complex is NOT an inv. of X.

Chainging the cell decomposition can change $C_n(X)$

There is an alg. trick to make this an inv.

$$C_i \longrightarrow H_i$$

$$\boxed{H_i(X) = \ker(d_{i-1}) / \operatorname{Im}(d_i)}$$

Terminology

An element in $H_k(X)$ is an equivalence class of k-cells.

i.e. if C is a k-cell

$[C]$ is some element in $H_k(X)$

the equivalence class of C

It may be trivial.

You can actually think of

- elements in H_1 as closed curves.
- elements in H_2 as surfaces

Facts:

1. Homology is a homotopy inv. and doesn't depend on the choice of cell decomposition.

2. Homology is also a homeo. inv. (by 1)

homeo $\Rightarrow$ homotopic

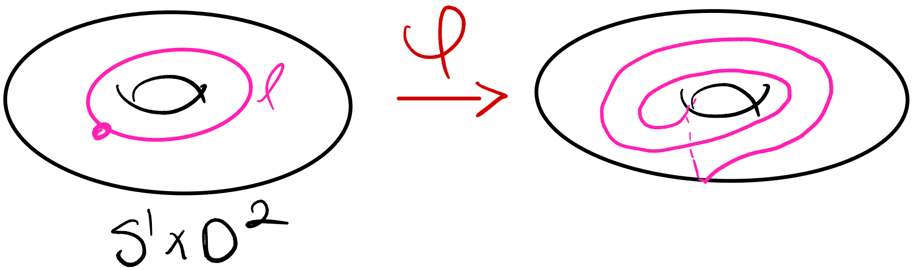
3 A homeomorphism (and homotopy equiv.)

$$f: X \to Y$$

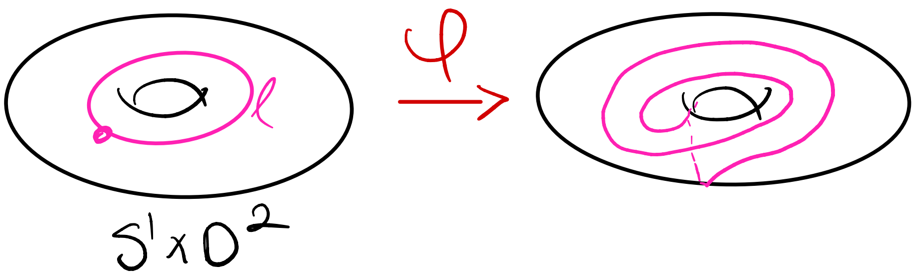
induces isomorphisms

$$f_*: H_n(X) \longrightarrow H_n(Y)$$

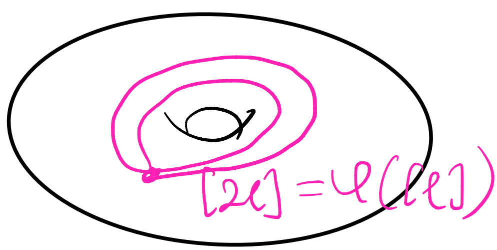
Ex: Is φ a homeomorphism?



Solution:



Is h.e.



$\varphi_*(\ell) = [2\ell]$

not an isom.

So, φ is NOT a homeo.