Intersection Form

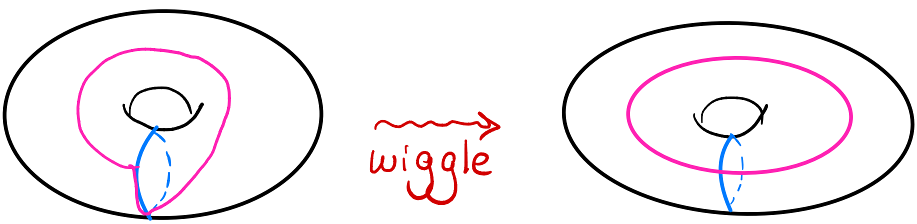
Intersection Number

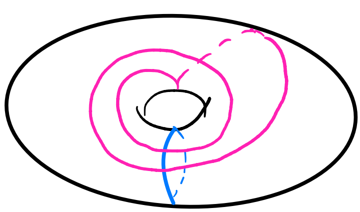
closed man: compact, $\partial X = \emptyset$

	1-man.	2-man.	3-man	4-man
closed	S^1	S^2, T^2, Σ_g	S^3	S^4
not closed	$\mathbb{R}^1$ $[0,1]$	D^2	B^3	B^4

Let X be a compact surface.

Any two closed curves on X can be wiggled so that they intersect in a finite set of pts

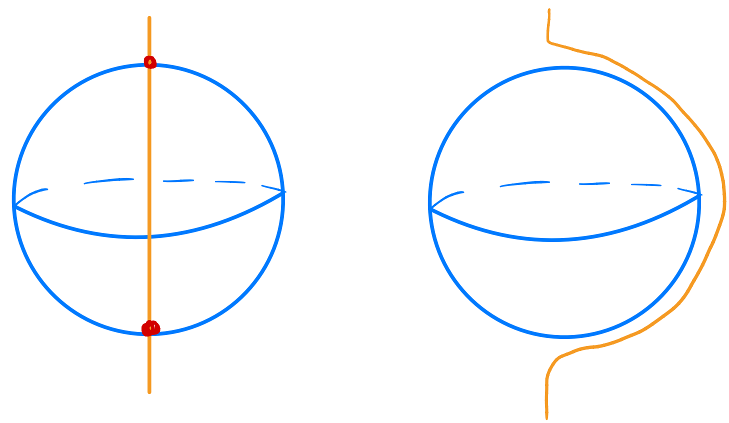




* $S \subset X^3$ a surface (2-dim.)
  $L \subset X$ 1-dim.

$\Rightarrow$ S and L can be wiggled so they intersect in a set of discrete pts.

Ex: $X = S^3$



* More generally

If A, B are closed subman. of M

and $\dim(A) + \dim(B) = \dim(M)$

$\Rightarrow$ A and B can be wiggled so that

they intersect at fin.ly many pts.

* In particular, if X is a 4-man.

and $S_1, S_2 \subset X$ closed surfaces,

$\Rightarrow$ S_1 and S_2 intersect at

fin.ly many pts.

Algebraic Intersection Number:

If $S_1, S_2 \subset X$ oriented

$\Longrightarrow$ each intersection pt. has a sign.

alg. int. $\# := \#(S_1 \cap S_2)$

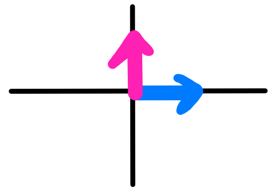
signed count of the intersection pts.

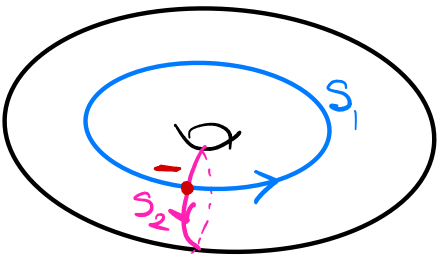
$+1$: If orientation of S_1 followed by orientation of S_2 agrees with the orientation on X

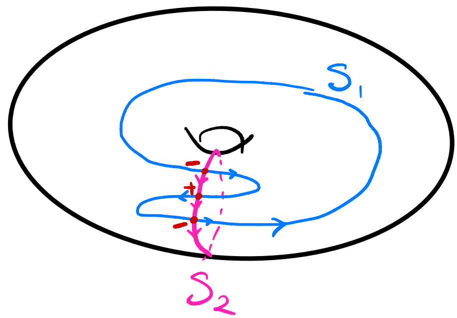
-1: Otherwise

Example:

Orientation on T^2







$\#(S_1 \cap S_2) = -1$

$\#(S_1 \cap S_2) = -1 + 1 - 1 = -1$

Self Intersection Number of a surface $S \subset X$

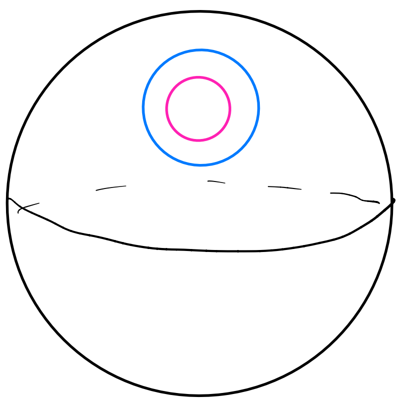
$\#(S, \tilde{S})$

$\tilde{S}$ = oriented push off of S.
(parallel)

Ex:

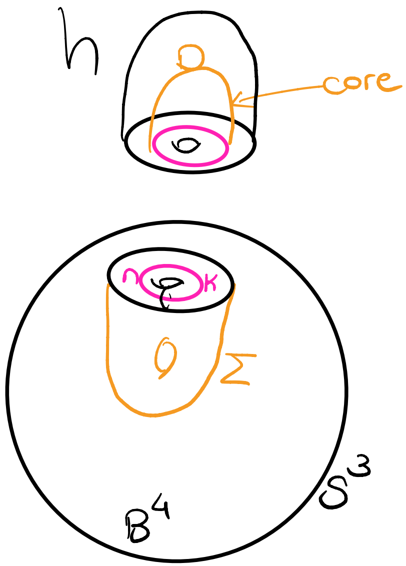
S intersects $\tilde{S}$ at one pt.

Ex:



self-linking = 0

Ex: 2-handles in 4-mans.:



Attach the 2-handle h to B^4 along K with framing n.

h: 2-handle with core D

$K \subset S^3$: a knot (attaching circle of h)

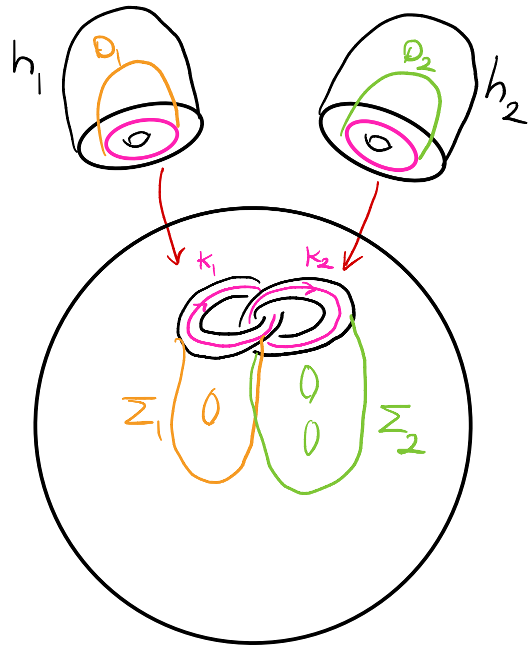
$\Sigma \hookrightarrow B^4$ embedded oriented surface

$\partial \Sigma = K$

$S = \Sigma \cap D$

$\Longrightarrow \quad \#(S \cap \tilde{S}) = n$

Ex:



h_i : 2-handles

$K_i \subset S^3$: knots (attaching circle of h_i)

$\Sigma_i \subset B^4$: surfaces with

$\partial \Sigma_i = K_i$

D_i : core of h_i,

S_i : $\Sigma_i \cap D_i$

$\Longrightarrow \#(S_1 \cap S_2) = lk(K_1, K_2)$

Intersection Form:

Let X be an oriented 4-man.

$$Q_X : H_2(X)/tor \times H_2(X)/tor \longrightarrow X$$

$$\begin{array}{ccc} H_2(X)/tor & & H_2(X)/tor \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a = [\Sigma_1] & & b = [\Sigma_2] \end{array}$$

$\Sigma_1, \Sigma_2 \subset X$: closed, oriented surfaces

$Q_X(a,b) = \#(\Sigma_1 \cap \Sigma_2)$ counted with sign

$Q_X(a,a) = \#(\Sigma_1 \cap \tilde{\Sigma}_1)$ counted with sign

Torsion subgroup: $T \leq G$ s.t.

$T = \{ g \in G : g \text{ has finite order} \}$

Remarks:

1. Any element in $H_2(X, \mathbb{Z})$ can be repr. by a smoothly embd., closed, oriented surface in X.

2. Q_X is a symmetric, bilinear form.

3. $Q_{X_1 \# X_2} = Q_{X_1} \oplus Q_{X_2}$

4. Given a basis for H_2

   Q_X can be repr. by a matrix.

$b_2^+(X), b_2^-(X) := \#\ \pm$ eigenvalues of Q_X

Invariants:

ⓐ rank(Q_M) =

$$\mathrm{rank}_{\mathbb{Z}}\left(H_2(M;\mathbb{Z})\right) = b_2(X) := b_2^+(X) + b_2^-(X)$$

ⓑ signature

$$\sigma(X) = \sigma(Q_X) = b_2^+(X) - b_2^-(X)$$

ⓒ parity: (even OR odd)

$$Q_X := \begin{cases} \text{even} & \text{if } Q_X(a,a) \equiv 0 \pmod 2 \quad \forall a \in H_2(X) \\ \text{odd} & \text{otherwise} \end{cases}$$

Defn: Q_X is called

positive definite: $Q_X(a,a) > 0 \quad \forall\, 0 \neq a \in H^2(X;\mathbb{Z})$

negative definite: $Q_X(a,a) < 0 \quad \forall\, 0 \neq a \in H^2(X;\mathbb{Z})$

definite : pos. or neg. definite

indefinite : Otherwise.

If X^4 is built from a 0-handle and n 2-handles without 1 and 3-handles

$\Longrightarrow H_2(X) \cong \mathbb{Z}^n$

h_i: core of the 2-handle with framing n_i.

Σ_i = Seifert Surface of K_i

$S_i = \Sigma_i \cup h_i$

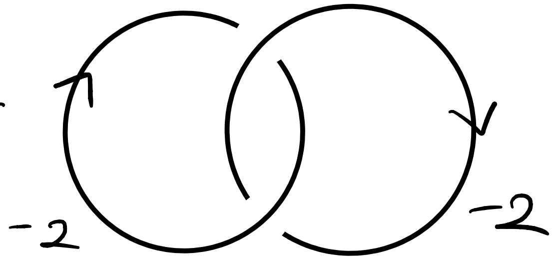
$\Longrightarrow \#(S_i \cap \tilde{S}_i) = n_i$

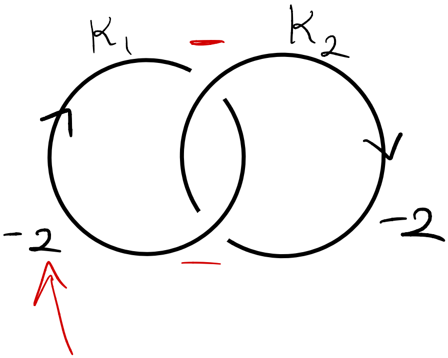
$\#(S_i \cap S_j) = lk(K_i, K_j)$

Moreover, if $\mathcal{B}_{H_2(X)} = \{[S_1], \ldots, [S_n]\}$

$$\Longrightarrow Q_X([S_i],[S_j]) = \begin{cases} lk(K_1, K_2) & i \neq j \\ n_i & i = j \end{cases}$$

Ex:

$X =$ 



$$\operatorname{lk}(K_1, K_2) = \frac{0-2}{2} = -1$$
$$\|$$
$$\operatorname{lk}(K_2, K_1)$$

$$\operatorname{lk}(K_1, K_1) = -2$$
$$\operatorname{lk}(K_2, K_2) = -2$$

$$Q_X = \begin{array}{cc} K_1 & K_2 \\ \multicolumn{2}{c}{\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}} \end{array} \begin{array}{c} K_1 \\ K_2 \end{array}$$

unimodular: Q_X is invertible over $\mathbb{Z}$.

i.e. $\det(Q_X) = \pm 1$

Fact:

X^4: closed $\Longrightarrow$ $\det(Q_X) = \pm 1$

non-deg. := invertible

i.e. $\det(Q_X) \neq 0$

Thm: [Donaldson]

X^4 : closed, oriented, smooth

Q_X : definite

$\Longrightarrow$ Q_X is diagonalizable.

i.e. $\exists$ a basis for $H_2(X)$ s.t. $Q_X = \pm I$

Conclusion:

X^4 : closed, oriented, smooth

Q_X : definite

$\Longrightarrow$ $\exists$ lattice isomorphism

$$\left(H_2(X)/_{tor}, Q_X \right) \longrightarrow \left(\mathbb{Z}^n, \pm I \right)$$

Thm: Q_X : indefinite,

ⓐ If Q_X is odd

$$Q_X = \begin{bmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{m} & 0 \\ 0 & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{n} \end{bmatrix} \sim m\langle 1\rangle \oplus n\langle -1\rangle$$

= up to change of basis

$m = b_2^+ \quad n = b_2^-$

Ex: $X = m\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2$ $\quad \text{rank} = b_2 = m+n$

$\sigma = m-n$

ⓑ If Q_X is even

$$Q_X = mE_8 \oplus nH \qquad m \in \mathbb{Z},\ n \in \mathbb{N}-\{0\}$$

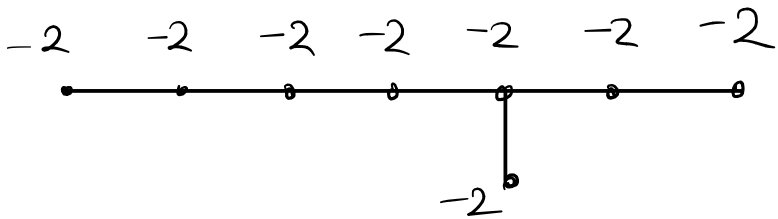
where $\sigma = -8m \qquad b_2 = 8|m| + 2|n|$

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ indef. even

$$\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$

unimodular
even
pos. definite
$\text{rank}(E_8) = 8 = \sigma(E_8)$

$Q_{S^2 \times S^2} = H$ $\qquad M_{E_8}$ is NOT smooth



Take 8 basis vectors and put

- -2 to the diagonal
- 1 if the corr. two vertices are connected
- 0 if they are not connected

Thm: Q_X : definite

$\Longrightarrow$ for a fixed rank

$\exists$ only fin.ly many different Q's.

(including $n\langle 1\rangle$, $m\langle -1\rangle$)

Thm: [Rokhlin]

X^4: cosc, simply-conn., Q_X even

($\Rightarrow$ spin)

$$\Rightarrow \sigma(X) \equiv 0 \pmod{16}$$

$$(\Rightarrow m = \text{even} = 2r)$$

Thm: [Donaldson]

X^4: cosc

Q_X + defn $\Rightarrow Q_X \sim m\langle 1\rangle$

$\quad$ − defn $\Rightarrow Q_X \sim n\langle -1\rangle$

Thm: [Donaldson]

$$\left.\begin{array}{l} Q_X : \text{even} \\ b_2^+(X) \leq 2 \end{array}\right\} \Rightarrow Q_X = H \ \text{OR} \ H \oplus H$$

10/8 -Thm: [Furuta]

$$\left.\begin{array}{l} Q_X : \text{even} \\ Q_X = 2r E_8 \oplus n H \end{array}\right\} \Rightarrow n \geq 2|r| + 1$$

Note: $\dfrac{b_2 - 8|m|}{2} \geq 2|r| + 1 \geq 2|r|$

$$\iff b_2 - 16|r| \geq 4|r| \iff b_2 \geq 20|r|$$

$$\iff b_2 \geq 20\frac{|\sigma|}{16} = \frac{10}{8}|\sigma|$$

11/8 Conjecture

Q_X : even

$$Q_X = 2r E_8 \oplus nH$$

$$\Longrightarrow n \geq 3|r|$$

Note: $\sigma = -8m = -16r$ $\qquad b_2 = 8|m| + 2|n|$

$$r = \frac{-\sigma}{16} \qquad n = \frac{b_2 - 8|m|}{2}$$

$$n \geq 3|r| \iff \frac{b_2 - 8|m|}{2} \geq 3|r|$$

$$\iff \frac{b_2 - |\sigma|}{2} \geq \frac{3}{16}|\sigma|$$

$$\iff b_2 - |\sigma| \geq \frac{3}{8}|\sigma| \iff b_2 \geq \frac{11}{8}|\sigma|$$

Thm: Freedman, Donaldson

X_1^4, X_2^4: smooth, closed, simply-conn.

$X_1 \underset{\text{homeo.}}{\simeq} X_2 \iff$ (1) $b_2(X_1) = b_2(X_2)$

(2) $\sigma(X_1) = \sigma(X_2)$

(3) X_1 and X_2 have the same parity

(i.e. both even OR odd)

Conclusion: X^4: cocs, simply-conn.

a) Q_X : odd $\Longrightarrow$

$X \simeq m\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}}^2 \quad m,n \geq 0$ OR

$(m = n = 0 \Rightarrow X \simeq S^4)$

b) Q_X : even $\Longrightarrow$

$X \simeq \ell K3 \# (n-3\ell)(S^2 \times S^2)$

$E(2) \qquad Q_{E(2)} = 2E_8 \oplus 3H$