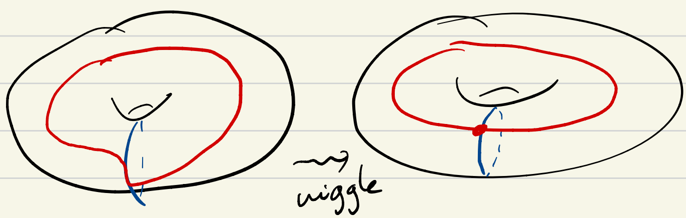
Intersection Numbers

Def: A manifold is closed is it is compact w/o boundary

	1-mfld	2-mfld	3-mfld	4-mfld
closed	S^1	S^2, T^2, Σ_g	S^3	S^4
not closed	$\mathbb{R}^1$ $[0,1]$		B^3	B^4

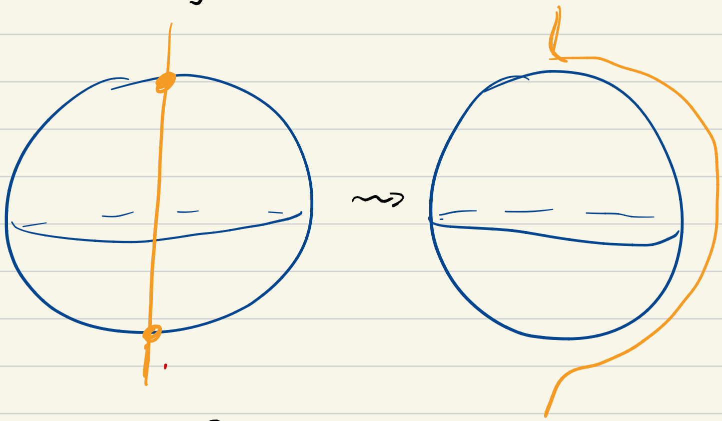
Let X be a compact surface

Any two closed curves on X can be wiggled so that they intersect in a finite set of point



Let X be 3-dim'l. If $S \subset X$ is a surface (2-dim'l) and $L \subset X$ is 1-dim'l, then S & L can be wiggled so they intersect in a set of discrete points

$X = S^3$



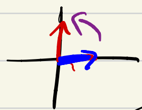
More generally, if A, B are closed submanifolds of M and $\dim A + \dim B = \dim M$, then A & B can be wiggled so that they intersect in finitely many points.

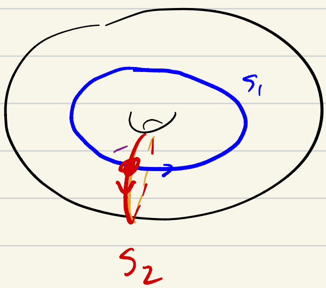
In particular, if X is a 4-manifold and $S_1, S_2 \subset X$ are closed surfaces, then S_1 & S_2 intersect in finitely many points.

If S_1, S_2, X are oriented, then each intersection point has a sign.

The (algebraic) intersection number of S_1 & S_2, $\#(S_1 \cap S_2)$ is the signed count of intersection points

Intersection point of S_1 & S_2 is $+1$ is orientation of S_1 followed by orientation on S_2 agrees with orientation on X
Otherwise, it is -1.

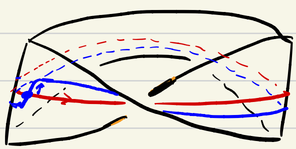
Ex: Orientation on T^2: 

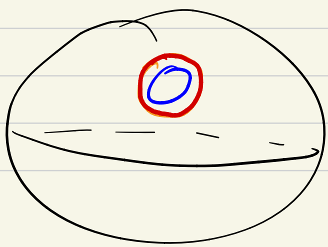


$\#(S_1 \cap S_2) = -1$

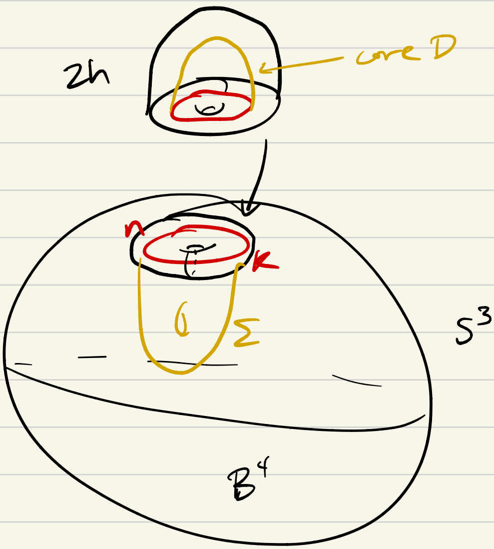
$\#(S_1 \cap S_2) = -1+1-1 = -1$

The self-intersection number of surface $S \subset X$ is $\#(S \cap \tilde{S})$, where $\tilde{S}$ is the oriented pushoff of S

Ex:  S intersects $\tilde{S}$ in one point

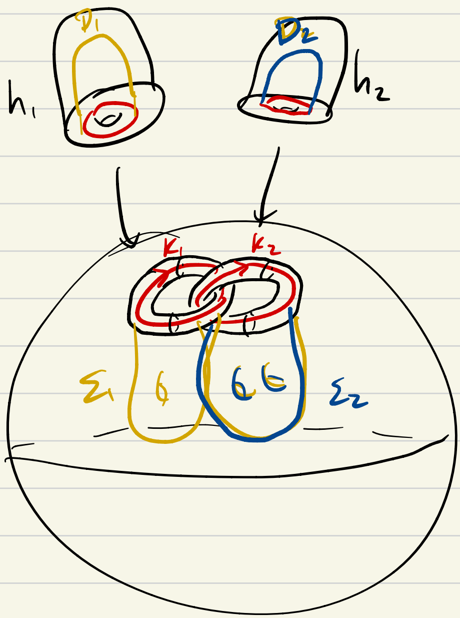
 Self-linking is 0

Ex: 2-handles in 4-manifolds



Let h be a 2-handle w/ core D, attached to B^4 along a knot K w/ framing n
Let $\Sigma \subset B^4$ be an embedded oriented surface with $\partial \Sigma = K$.
Let $S = \Sigma \cup D$.

Then $\#(S \cap \tilde{S}) = n$



Let h_1, h_2 be two 2-handles As above, let K_i be the attaching circle of h_i, let $\Sigma_i \subset B^4$ be a surface w/ $\partial \Sigma_i = K_i$, and let D_i = core of h_i.

Let $S_i = \Sigma_i \cup D_i$.

Then $\#(S_1 \cap S_2) = lk(K_1, K_2)$

Homology

The homology of a 4-manifold can be computed using the handlebody diagram

Fact: Any element in $H_2(X;\mathbb{Z})$ can be represented by a smoothly embedded closed oriented surface in X

Let X be an oriented 4-mfld

Def: The intersection form Q_X on X is a symmetric bilinear form

$$Q_X : H_2(X;\mathbb{Z})/_{\text{Torsion}} \times H_2(X;\mathbb{Z})/_{\text{Torsion}} \longrightarrow \mathbb{Z} \quad \text{such that}$$

If Σ_1, Σ_2 are closed oriented surfaces in X, then

$$Q_X([\Sigma_1],[\Sigma_2]) = \#(\Sigma_1 \cap \Sigma_2) \quad \text{counted with sign}$$

$$Q_X([\Sigma_1],[\Sigma_1]) = \#(\Sigma_1 \cap \tilde{\Sigma}_1), \quad \text{where } \tilde{\Sigma}_1 = \text{oriented pushoff of } \Sigma_1$$

X is called positive-definite if Q_X is positive-definite
and negative-definite if Q_X is negative-definite

Given a basis for $H_2(X)$, we can write down an explicit matrix for Q_X.

If X is a 4-manifold built from a 0-handle and n 2-handles w/o 1,3 handles

then $H_2(X) \cong \mathbb{Z}^n$

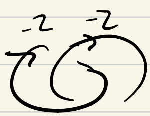
Let h_i denote the core of the i^{th} 2-handle w/ framing a_i and Σ_i denote a seifert surface for K_i

Let $S_i = \Sigma_i \cup h_i$

then $\#(S_i \cap \tilde{S}_i) = a_i$ and $\#(S_i \cap S_j) = lk(K_i, K_j)$

Moreover, $\{[S_1], \dots, [S_n]\}$ forms a basis for $H_2(X)$

Then $Q_X([S_i],[S_j]) = \begin{cases} lk(K_i,K_j) & i \neq j \\ a_i & i = j \end{cases}$

Ex: $X =$ 

$Q_X = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

If $\det Q_X \neq 0$, then Q_X is a nondegenerate symmetric bilinear form.

Fact: If X is a closed, oriented 4-manifold, then $\det Q_X = \pm 1$

Thm (Donaldson):

Let X be a closed, oriented, smooth, positive or negative definite 4-manifold. Then it's intersection form Q_X is diagonalizable.

(ie. $\exists$ a basis for $H_2(X)$ so that Q_X has matrix $\pm I$).

$\Rightarrow$ If X is a closed, oriented, smooth, pos/neg-def 4-mfld, then $\exists$ lattice isomorphism $(H_2(X)/_{tor}, Q_X) \to (\mathbb{Z}^n, \pm I)$.