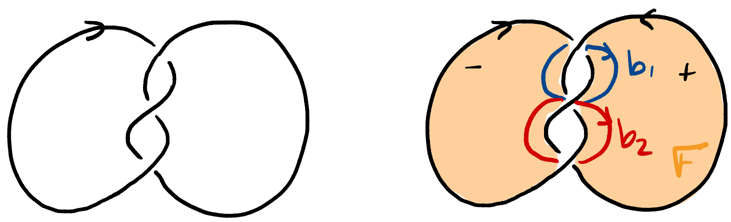
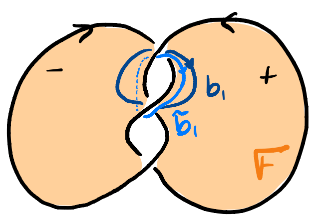
# Invariants and Sliceness Obstructions

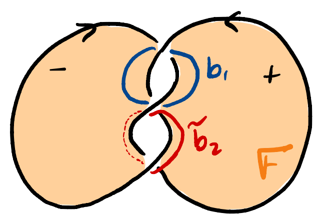
Given an oriented link $L$ and a Seifert surface $F$ for $L$, pick a collection of oriented curves $\{b_1, \dots, b_n\}$ on $F$, that "surround the holes." (this is a basis for $H_1(F) \cong \mathbb{Z}^{2g}$)




Let $V$ be the $2g \times 2g$ matrix whose $(i,j)$-th entry is $\ell k(b_i, \tilde{b}_j)$, where $\tilde{b}_j$ is a pushoff of $b_j$ off $F$ in the positive direction. $V$ is called a Seifert Matrix.

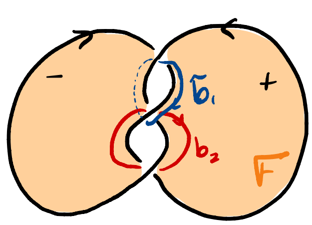



$\ell k(b_1, \tilde{b}_1) = -1$

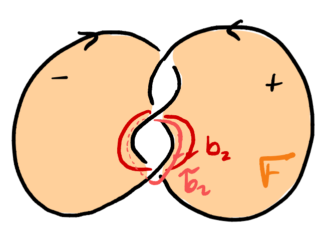



$\ell k(b_1, \tilde{b}_2) = 0$

$$\Rightarrow V = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$




$\ell k(b_2, \tilde{b}_1) = 1$

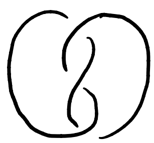



$\ell k(b_2, \tilde{b}_2) = -1$

**Def:** The determinant of $L$ is $\det(L) := \det(V+V^T)$

**Def:** The signature of $L$ is

$$\sigma(L) := \sigma(V+V^T) = \#\text{ positive eigenvalues} - \#\text{ negative eigenvalues}$$

**Ex:** For $K =$ ,

$$V+V^T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det K = \det \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = 3$$

$$\sigma(K) = \sigma\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\right) = -2$$ (since it is negative definite)

(Check with KLO)

**Facts:** 1) det and $\sigma$ are invariants of links.

That is, if $\det L_1 \neq \det L_2$ or $\sigma(L_1) \neq \sigma(L_2)$, then $L_1 \neq L_2$.

2) If $L$ is $\chi$-slice, then $|\det L|$ is a square.

3) If $K$ is slice, then $\sigma(K) = 0$ ($K$ a knot)

If $L$ is $\chi$-slice and bounds an orientable surface with $\chi = 1$, then $\sigma(L) = 0$.

**Ex:** The trefoil is not slice.

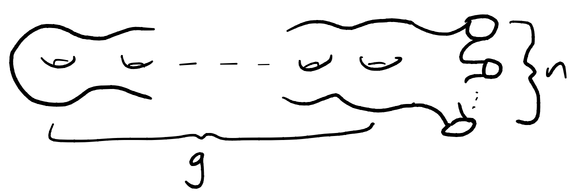
By the above facts, det and $\sigma$ are slicеness obstructions. More precisely:

- If $|\det L|$ is not a square, then $L$ is not $\chi$-slice
- If $\sigma(K) \neq 0$, then $K$ is not slice

For links with more than one component, we can also obstruct $\chi$-sliceness geometrically by considering the classification of surfaces and linking numbers.

Recall the classification of surfaces:

- An oriented surface is homeomorphic to

$\Sigma_g^n =$ 

genus $g$ surface with $n$ boundary components

$$\chi(\Sigma_g^n) = 2 - 2g - n \quad (g, n \geq 0)$$

Note: $\Sigma_0^1 = D =$ disk, $\Sigma_0^2 = A =$ annulus

- A nonorientable surface is homeomorphic to

$$P_k^n = \#_k P - \left(\bigcup_n D_i\right) \quad (n \geq 0,\ k \geq 1)$$

$$\chi(P_k^n) = 2 - k - n$$

Note: $P_1^1 = M =$ Mobius band.

Ex: What are the surfaces with $\chi=1$ and 2 boundary components?

Let $F$ be a surface with $\chi(F)=1$ and $\partial F = L_1 \sqcup L_2$.

If $F$ is connected, then either

$\chi(F) \leq -2g \leq 0$ (if $F$ is orientable)

$\chi(F) \leq -K \leq -1$ (if $F$ is unorientable)

which are both impossible.

Thus $F$ is disconnected and has 2 components

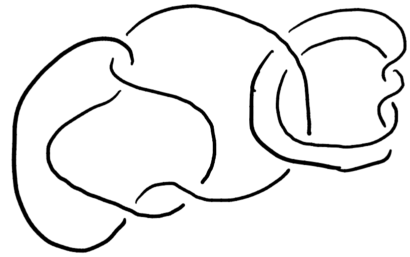
$F = F_1 \sqcup F_2$ with $\partial F_1 = L_1$, $\partial F_2 = L_2$

Now $\chi(F) = \chi(F_1) + \chi(F_2) = 1$ and $\chi(F_1)$, $\chi(F_2) \leq 1$

Thus we may assume, without loss of generality that $\chi(F_1)=1$ and $\chi(F_2)=0$.

By considering the $\chi$ formulas with $n=1$, we see that $F_1$ is a disk and $F_2$ is a Mobius band

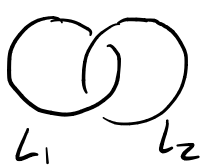
So the only surface with $\chi=1$ and two boundary components is Disk $\sqcup$ Mobius band.

Ex: Show that $L =$  is not $\chi$-slice

If $L$ is $\chi$-slice, then $L = \partial(\text{Disk} \sqcup \text{Mobius band})$ and so one of the components is slice. However, we can check that neither are slice by calculating det and $\sigma$.

Fact: Let $F_1$ and $F_2$ be disjoint embedded surfaces in $B^4$ with one boundary component.

Then $lk(\partial F_1, \partial F_2) \equiv 0 \bmod 2$.

Ex: Show that $L =$  is not $\chi$-slice

If $L$ is $\chi$-slice, then it bounds $\text{Disk} \sqcup \text{Mobius band}$.

By the above fact, $lk(L_1, L_2) \equiv 0 \bmod 2$

But $lk(L_1, L_2) = \pm 1$ (depending on orientation),

so $L$ is not $\chi$-slice.

# Lattice Embedding Obstruction

The main obstruction we employ is a lattice theoretic obstruction.

In short, if $L$ is $\chi$-slice with $\det L \neq 0$ and $\exists$ an associated negative-definite symmetric bilinear form $Q$, then $\exists$ a lattice embedding $\varphi : (\mathbb{Z}^n, Q) \longrightarrow (\mathbb{Z}^n, -I)$

Moreover, under an additional hypothesis related to $Q$, $\varphi$ is cubiquitous.

Over the next week, we will see how this lattice embedding comes to be.