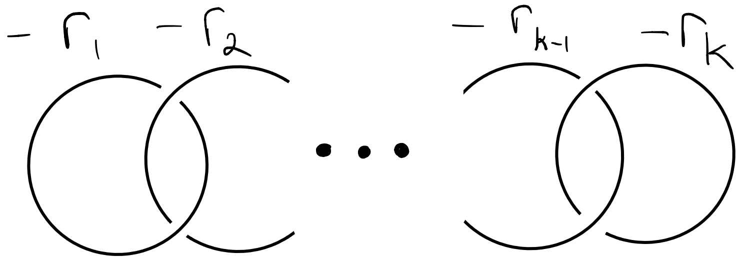
Kirby Calculus

- Slam Dunk
- Rolfsen Twist
- Blow-up
- Blow-down
- Handle Slide

Ex: Lens Spaces:

$X^4 =$ 

$$L(p,q) = S^3 \big/ (z_1, z_2) \sim \left(e^{2\pi i/p} z_1,\ e^{2\pi i q/p} z_2\right)$$

$$\pi_1\big(L(p,q)\big) = \mathbb{Z}_m$$

$$\partial X^4 = L(p,q)$$

Continued Fraction:

$$[r_1, r_2, \dots, r_k] = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}} = \frac{p}{q}$$

$$r_i \geq 2$$

$$\Longrightarrow \partial X = L(p,q)$$

Fact: $L(p,q) = S^3_{-p/q}(U)$

Kirby Moves / Surgery Moves

Goal:

1. Reduce # link components if possible.

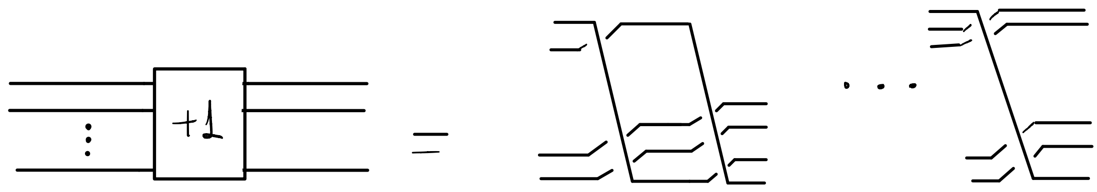
2. Change coefficients which are fractions to be integral or even ± 1

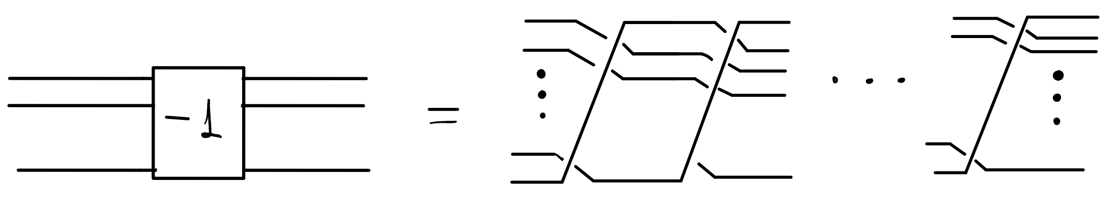
Note: Depending on your goal often we want either 1 OR 2

Remark:

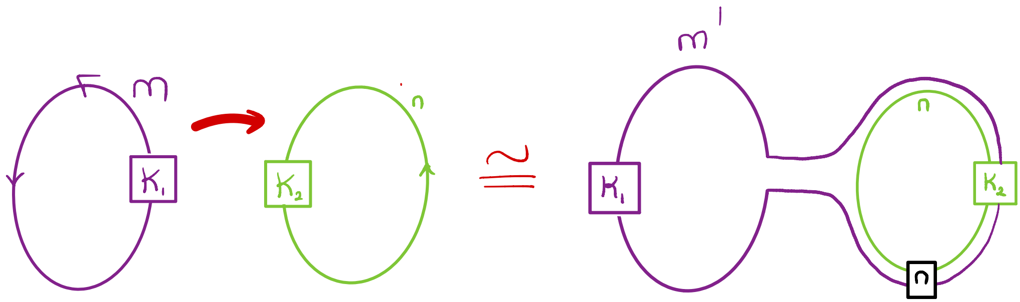
1. For us, $lk(K_1, K_2)$ will usually be ± 1.

2. a full-twist on k-strands





2-Handle Slide



$n \in \mathbb{Z}$

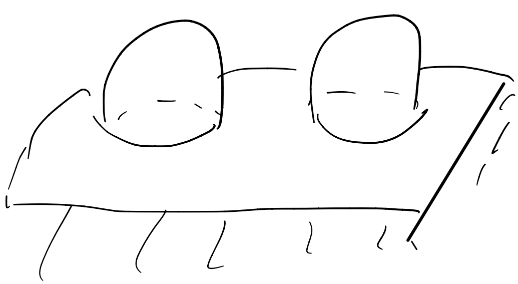
K_1 and K_2 can link

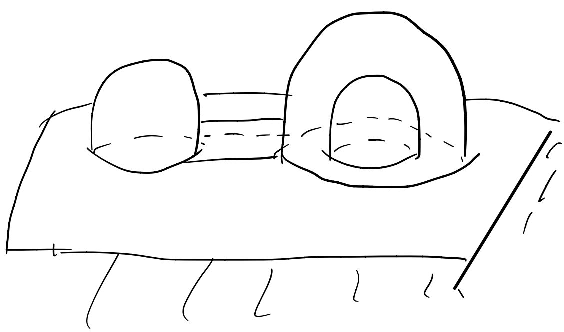
$m \in \mathbb{Q} \cup \{\infty\}$

$$m' = m + n + 2\,lk(K_1, K_2)$$

If one arrow reversed

$$m' = m + n - 2\,lk(K_1, K_2)$$

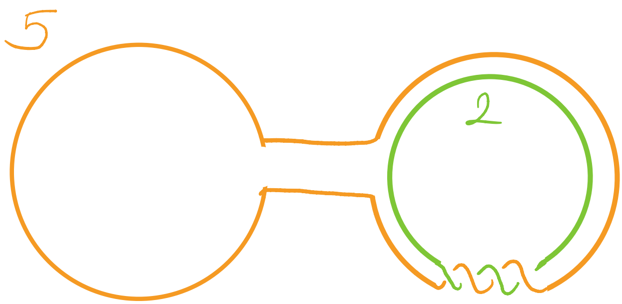




Ex:

$lk(K_1, K_2) = 0$ so $m' = n + m + 0 = 2 + 3 = 5$

$=$

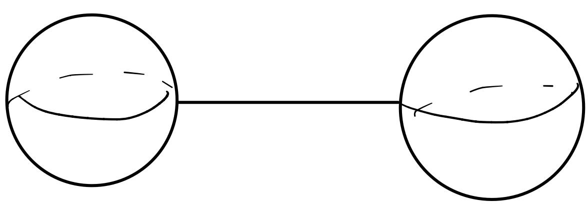


Handle Cancelation

Canceling handle pair

A $(k-1)$ handle and k-handle can be canceled if the attaching sphere of the latter intersects the belt sphere of the former transversely in a unique point.

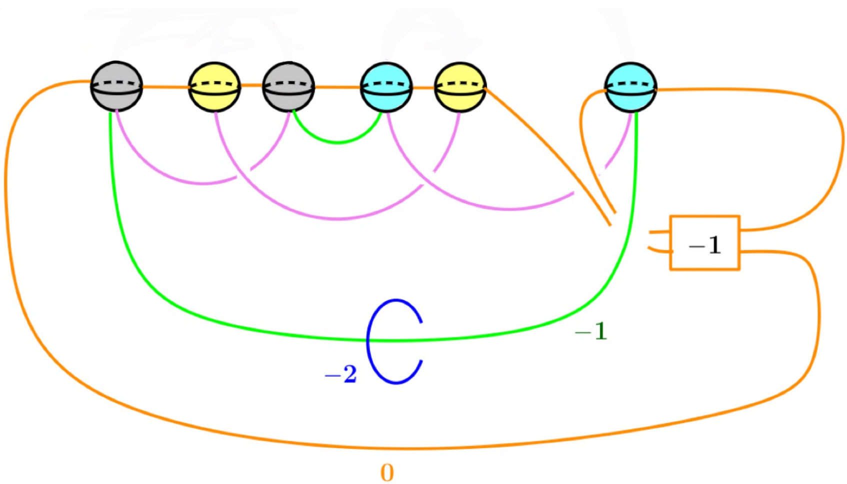
$k = 2$:



erase both handles

You can cancel this

Ex:



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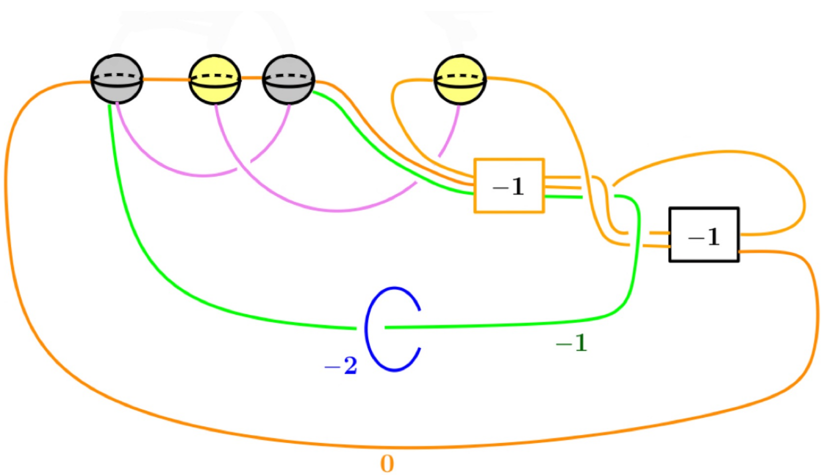
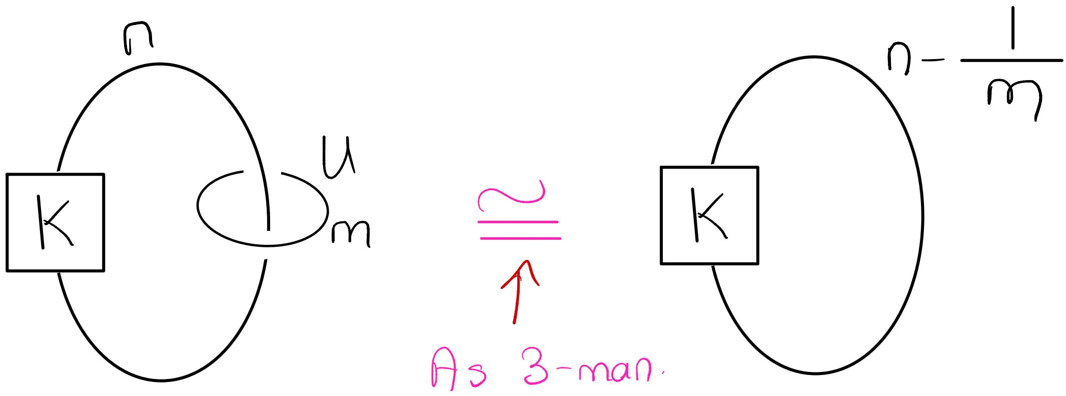


Slam dunk



$n \in \mathbb{Z}$

$m \in \mathbb{Q} \cup \{\infty\}$

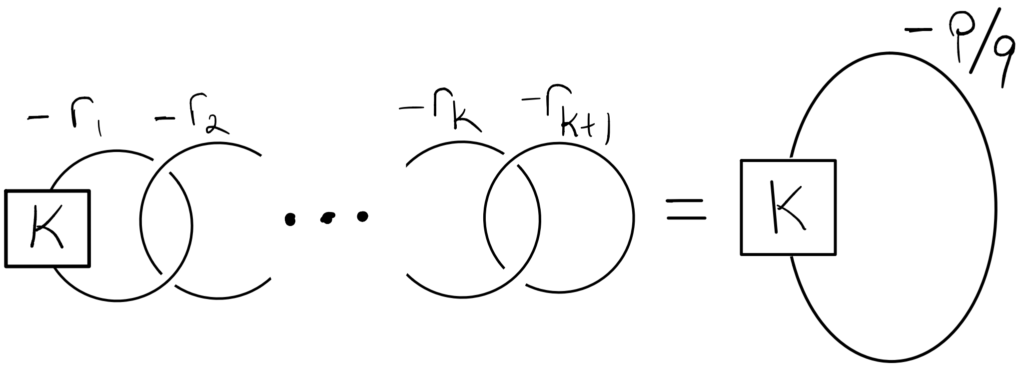
Remarks:

1. This has NO 4-dim. interpretation since coefficients won't be integral on the right hand side.

2. This gives two different surgery description of the same manifold.

Corollary:

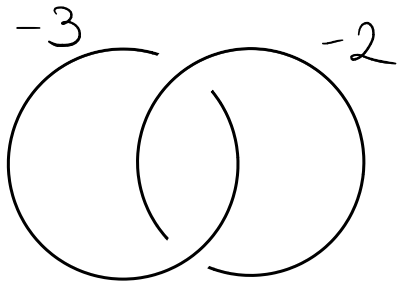
Let $(p,q)=1$

$\Rightarrow$ 

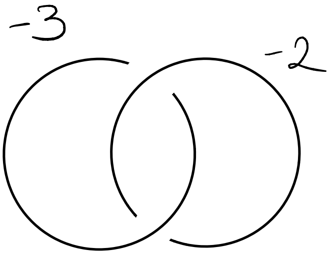
where $$\frac{p}{q} = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}}$$

Proof: Just slam-dunk $(k-1)$-times

Ex: X^4

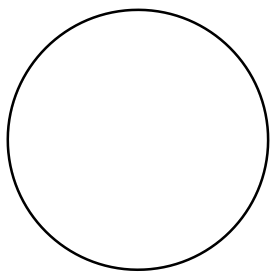


Now, thinking that as a surgery diagram for the boundary:



$\xrightarrow{\text{Slam-dunk}}$

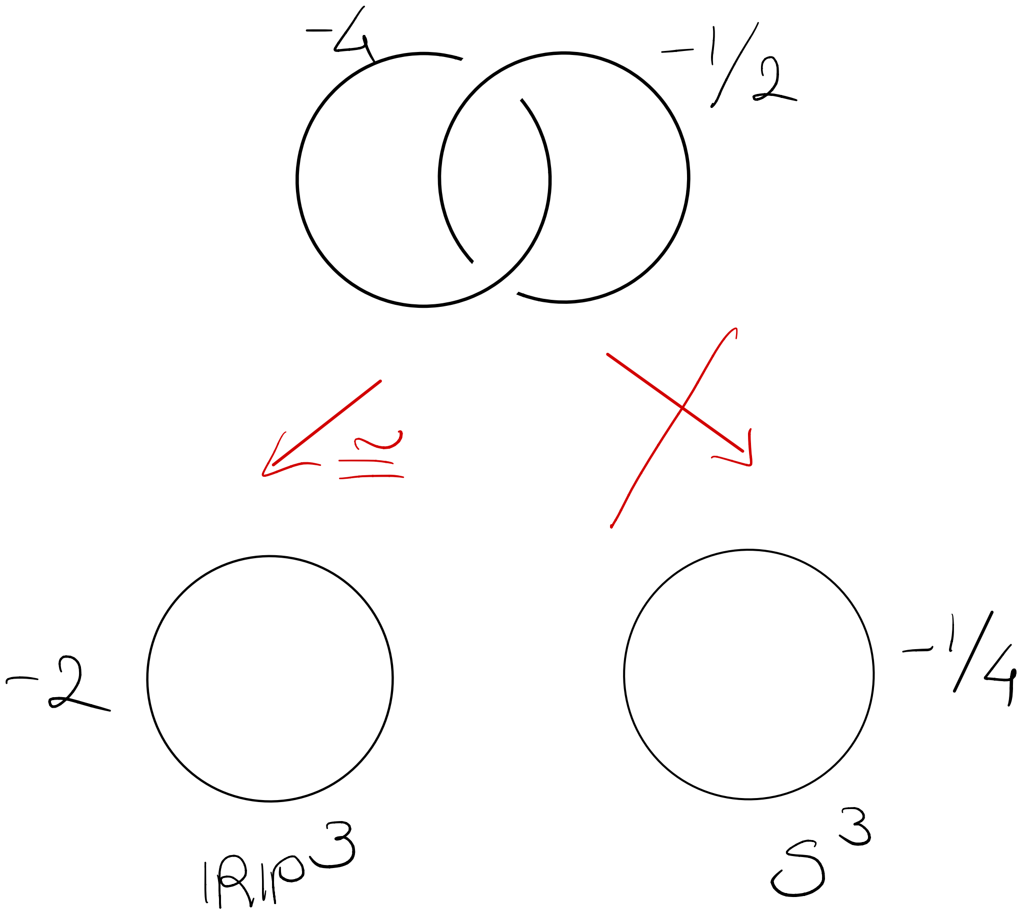
$$-3-\left(\frac{1}{-2}\right) = -5/2$$



$S^3_{-5/2}(U)$

No longer has any 4-dim. meaning.

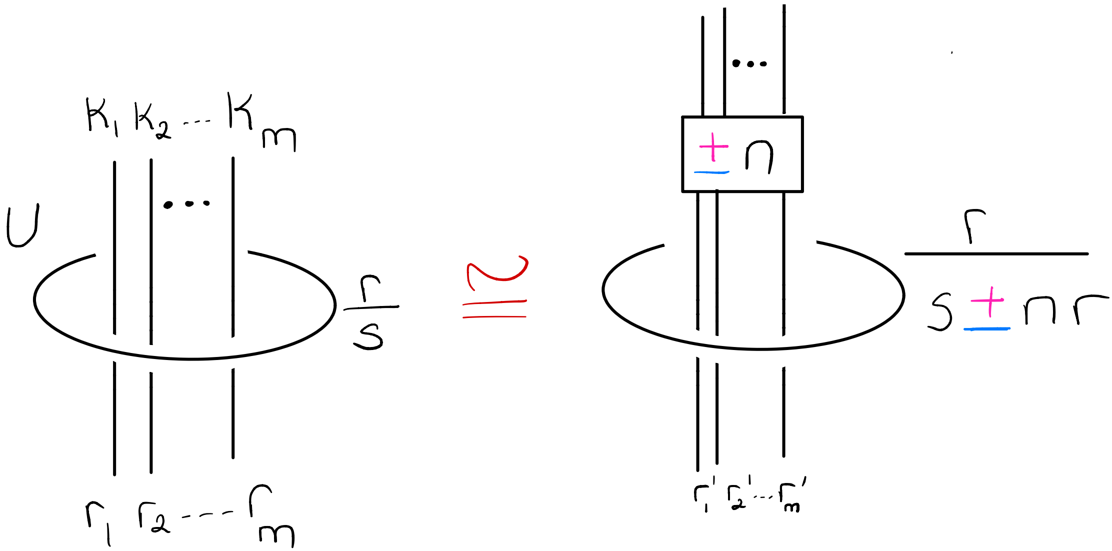
Warning about Slam dunk



$$-4-\frac{1}{-1/2}=-2 \qquad\qquad -\frac{1}{2}-\frac{1}{-4}=-\frac{1}{4}$$

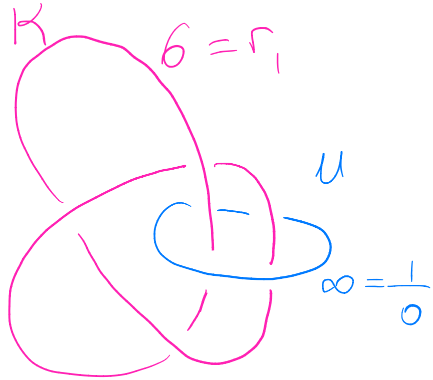
Recall: $n \in \mathbb{Z}$ $\qquad -1/2 \notin \mathbb{Z}$

Rolfsen Twist

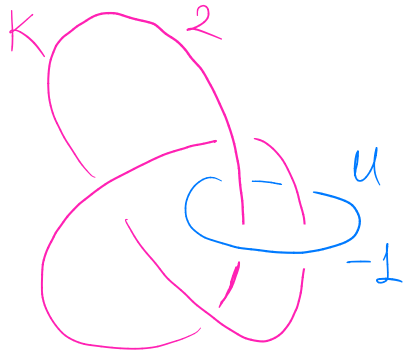


$$r_i' = r_i \pm n\left[lk(K_i, U)\right]^2$$

Ex:



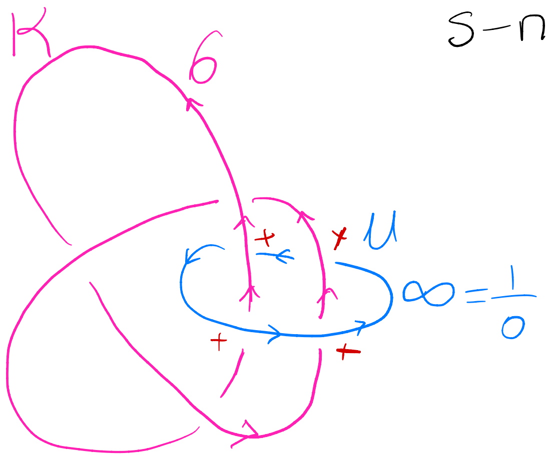
$\cong$



$\infty = \frac{1}{0} = \frac{r}{s}$ $\quad r = 1 \quad s = 0$

$\boxed{\text{Choose } n = 1}$

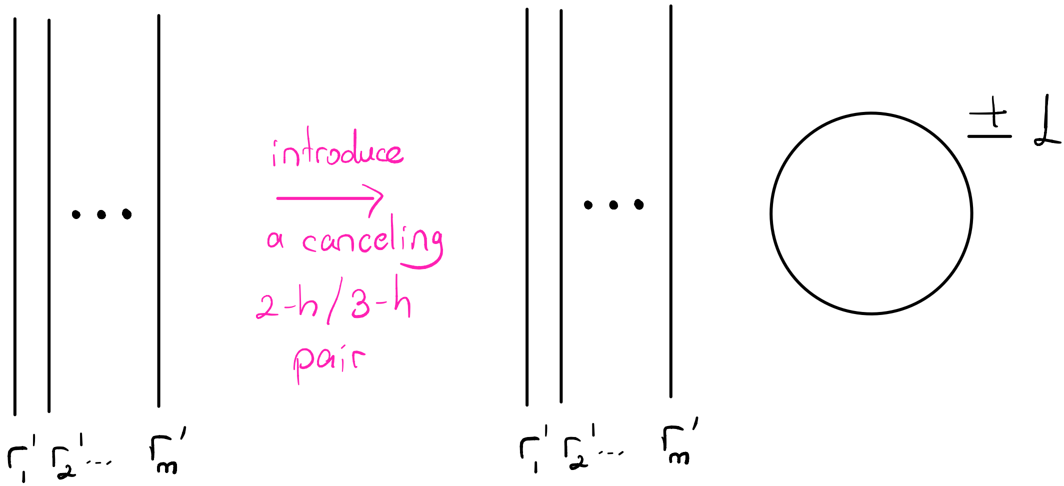
$$\frac{r}{s - nr} = \frac{1}{0 - 1 \cdot 1} = -1$$



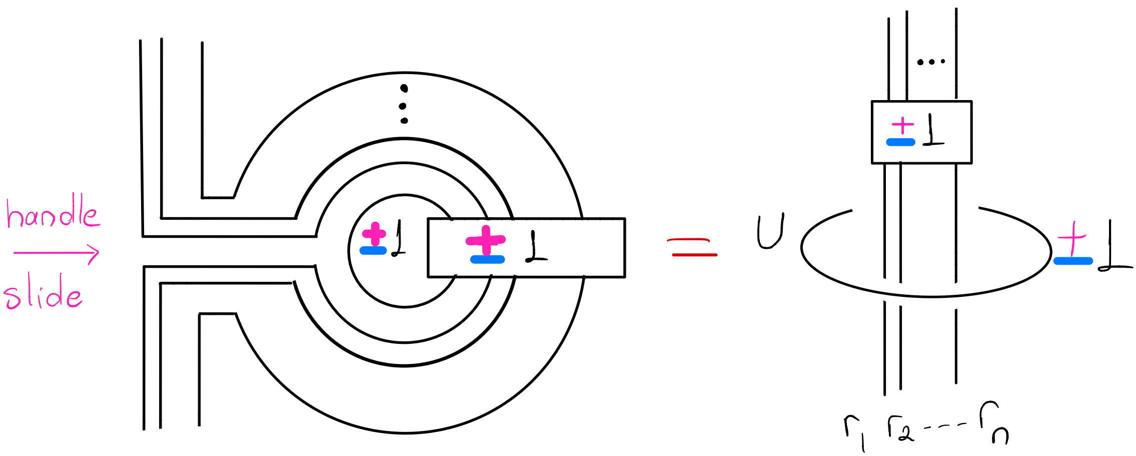
$lk(K, U) = \mp 2$

$$r_1' = r_1 - 1\, lk(K_1, U)^2 = 6 - 1 \cdot 4 = 2$$

Blow-Up

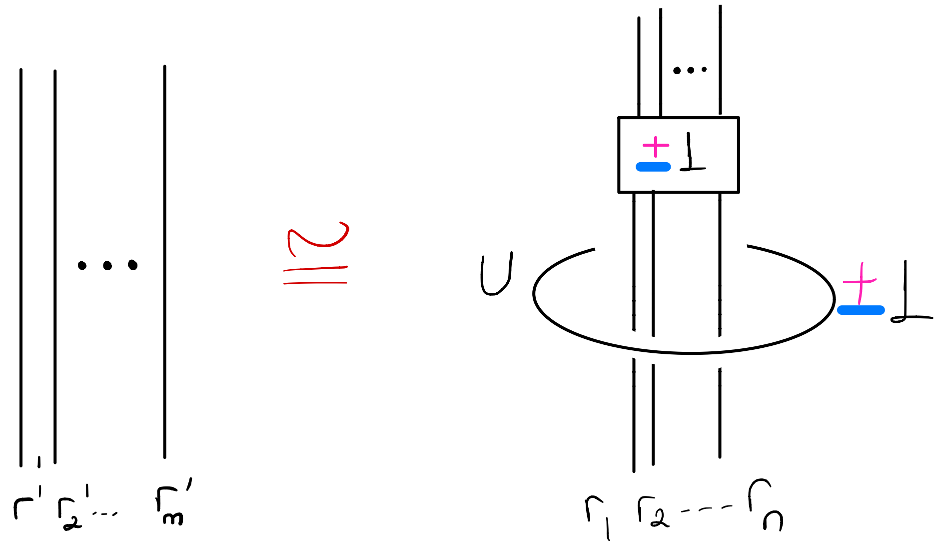


Γ_1' Γ_2' ... Γ_m'
introduce
a canceling
2-h/3-h
pair
Γ_1' Γ_2' ... Γ_m'
± 1



handle
slide
± 1
± 1
=
U
± 1
± 1
Γ_1 Γ_2 --- Γ_n

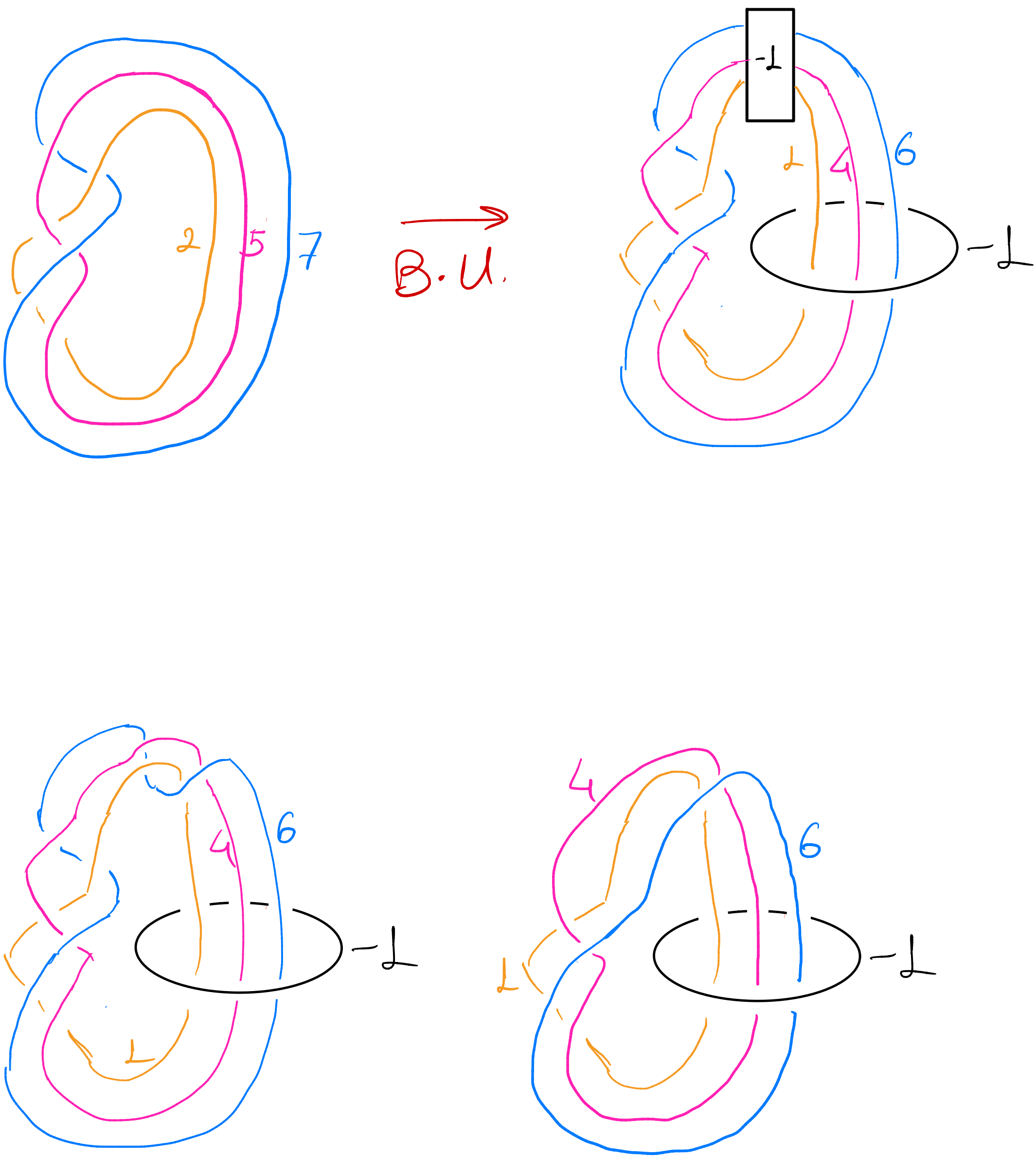
Blow-Up

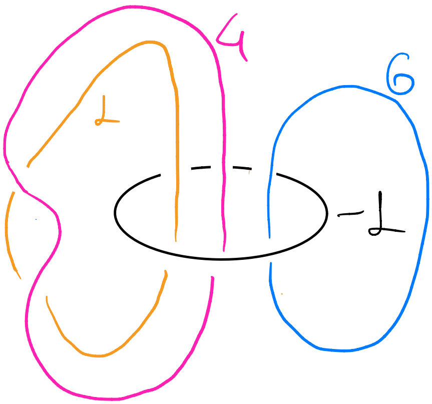


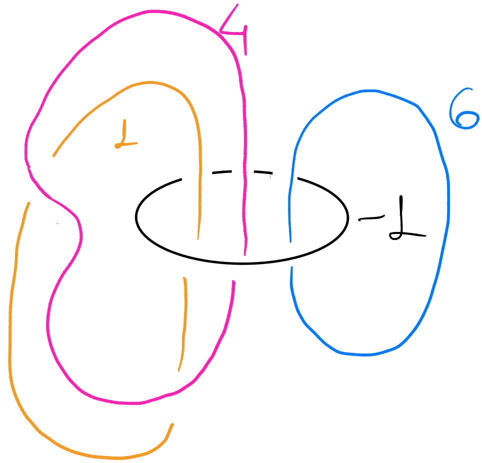
A **blow-up** of a surgery diagram is the **addition** of an unknotted component, unlinked from the rest of the diagram, with surgery coeff. ± 1

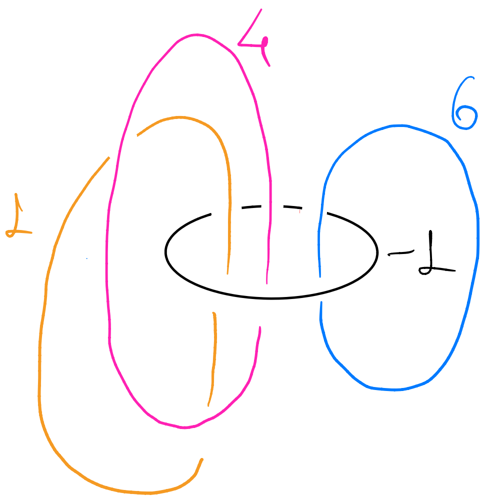
$$r_i' = r_i \mp [lk(K_i, U)]^2$$

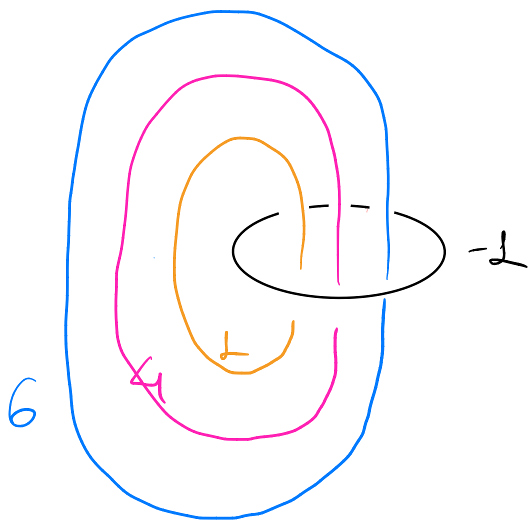
Ex:



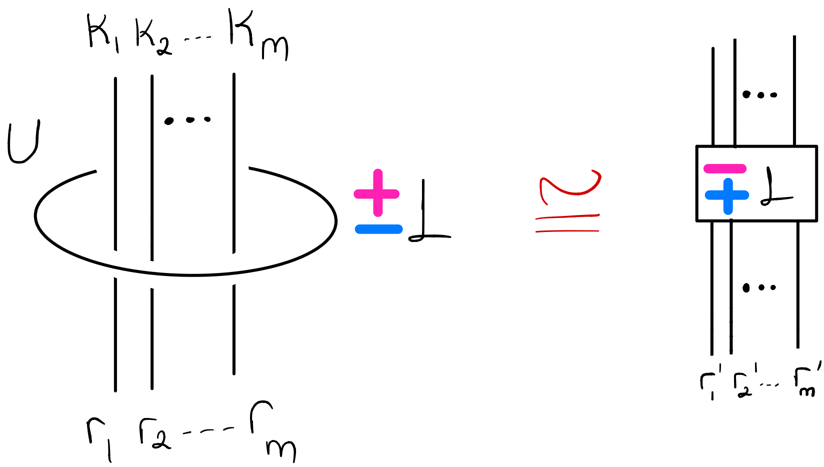

4
6
1
-1


4
6
1
-1


4
6
1
-1


-1
1
4
6

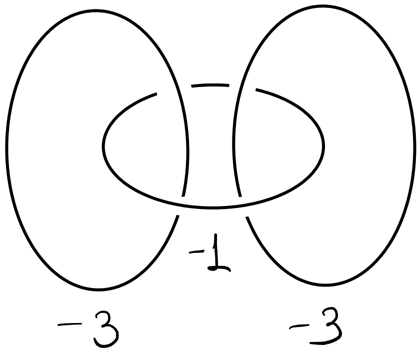
Blow-Down



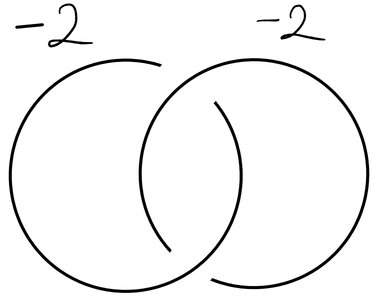
A blow-down of a surgery diagram is the removal of an unknotted component, unlinked from the rest of the diagram, with surgery coeff. ± 1

$$r_i' = r_i \mp [lk(K_i, U)]^2$$

Ex:



$\underset{\text{B.D.}}{\cong}$



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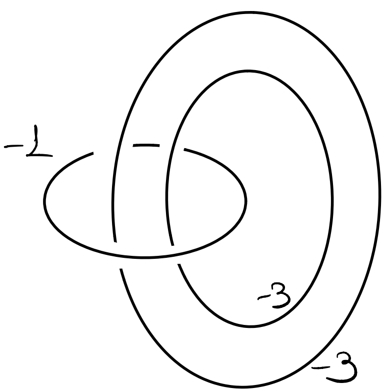
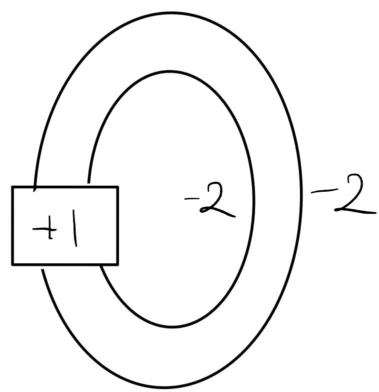
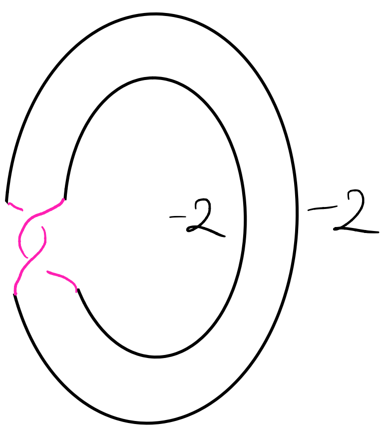


$\underset{\text{B.D.}}{\cong}$





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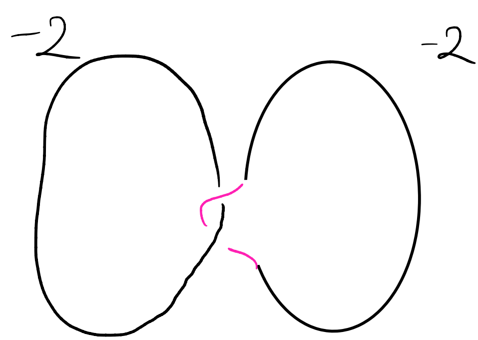
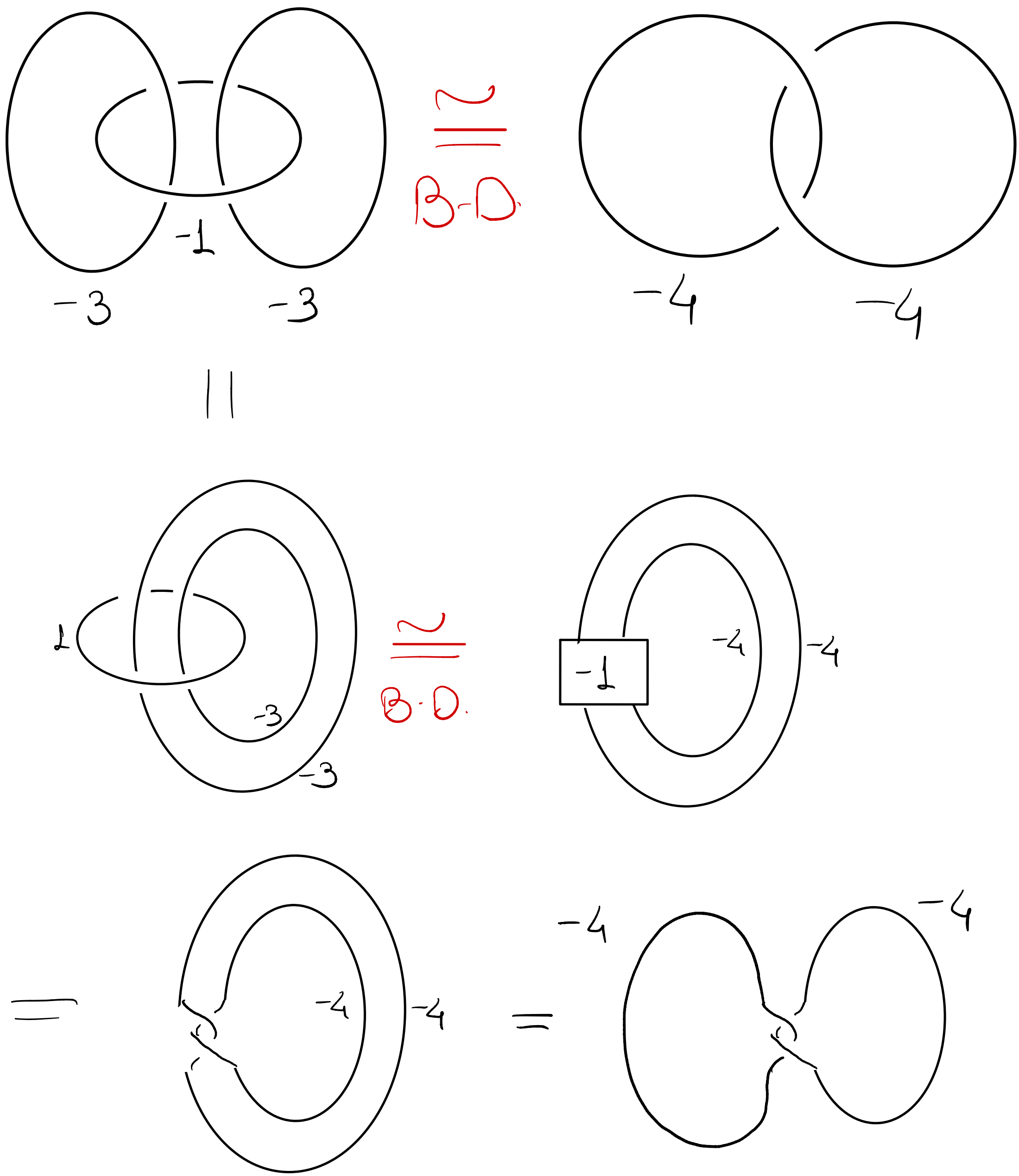
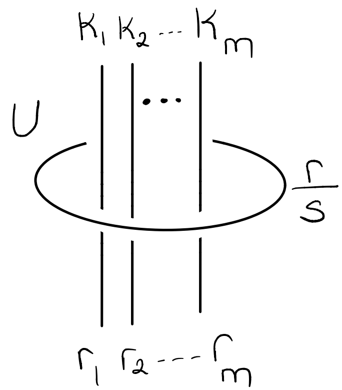


Ex :



Note: Rolfsen twist include blow-down as a special case.



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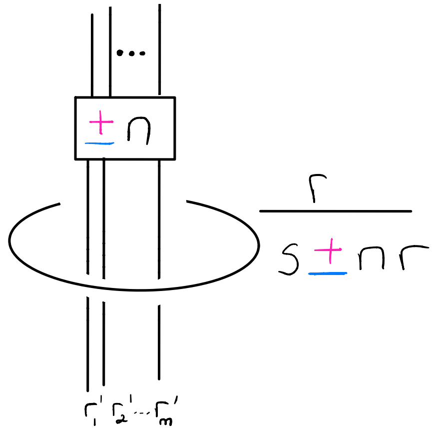
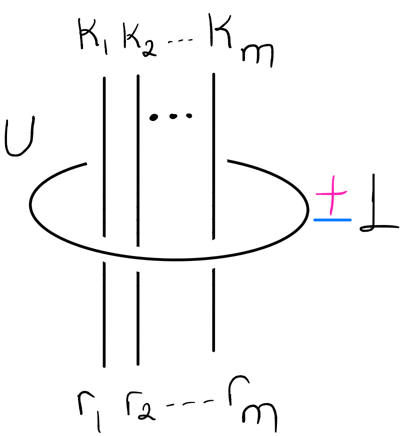


$\frac{r}{s} = \pm 1$

$r = 1, \; n = 1$

$s = -1 \Rightarrow \boxed{1} \quad \frac{r}{s+nr} = \frac{1}{-1+1} = \frac{1}{0}$

$s = 1 \Rightarrow \boxed{-1} \quad \frac{r}{s+nr} = \frac{1}{1-1} = \frac{1}{0}$



$\cong$
B.D