- Knots
- Links
- Spanning Surfaces

Knot in M^3:

$K^1 \subset M^3$: connected 1-dim. $\underbrace{\text{submanifold}}$

- embedded
- $\partial K = \emptyset$

A knot is an embedding

$$S^1 \hookrightarrow M^3$$

Remark:

Manifolds $\longleftrightarrow$ homeomorphisms

Knots $\longleftrightarrow$ isotopy

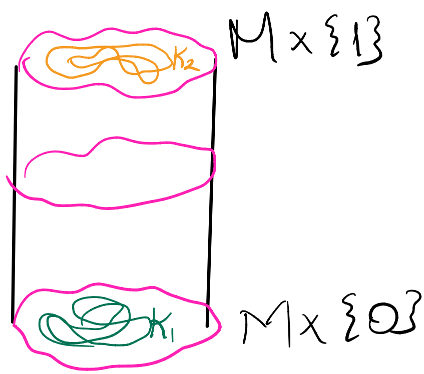
isotopy := continuous distortion/deformation

ambient isotopy of K_1 to K_2

$$F : M \times [0,1] \longrightarrow M \quad \text{s.t.}$$

1. F is continuous
2. $F_0(K_1) = K_1$ (actually $F_0 = id$)
3. $F_1(K_1) = K_2$
4. $F_t : M \to M$ homeo. $\forall t \in [0,1]$

Notation: $F_t : M \times \{t\} \to M$



Defn: Two knots K_1 and K_2 are *isotopic* if

$\exists\, \varphi : S^3 \longrightarrow S^3$ homeo.

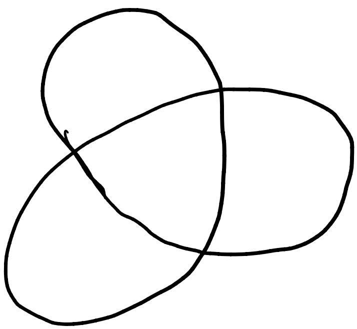
s.t. $\varphi(K_1) = K_2$

Diagrams:

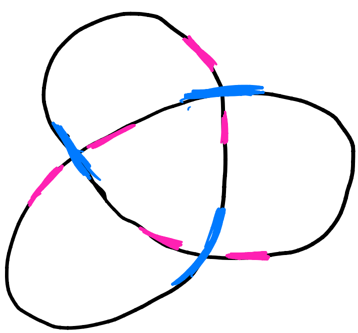
How to express knots in $\mathbb{R}^3 / S^3$

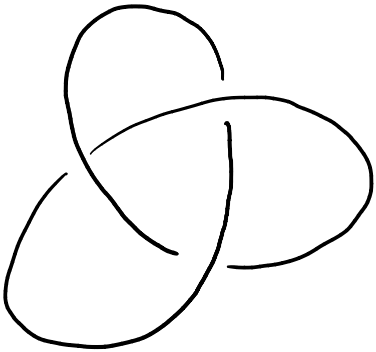
project K onto $\mathbb{R}^2 / S^2$ in $\mathbb{R}^3 / S^3$

"good projection"

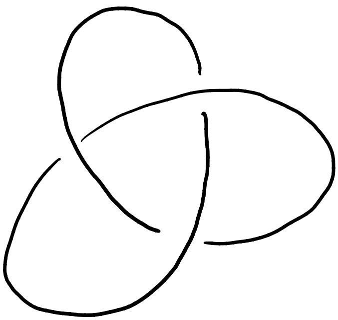
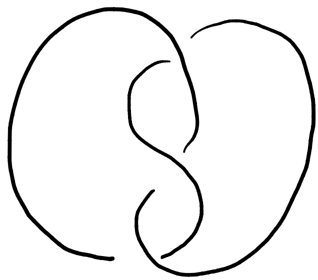


$\downarrow$ remember crossing info



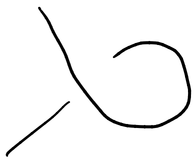


trefoil

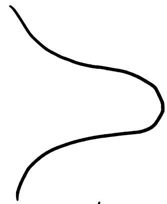
Q:  $\simeq$ isotopic ? 

Reidemeister Moves:

1

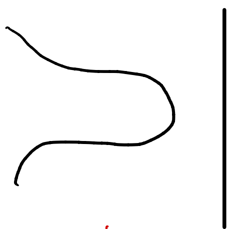


twist

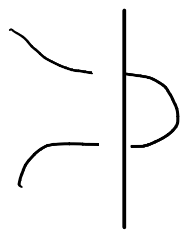


untwist

2

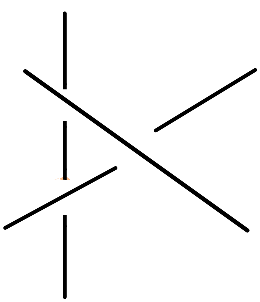


unpoke

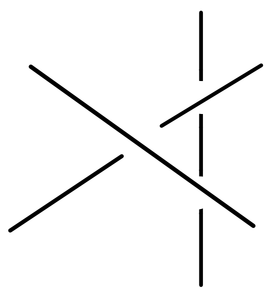


poke

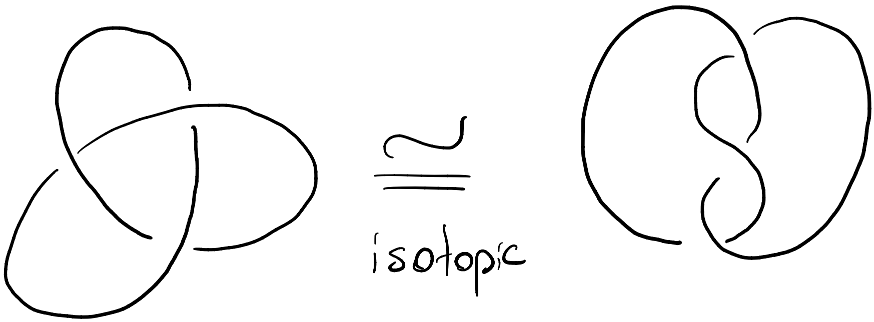
3

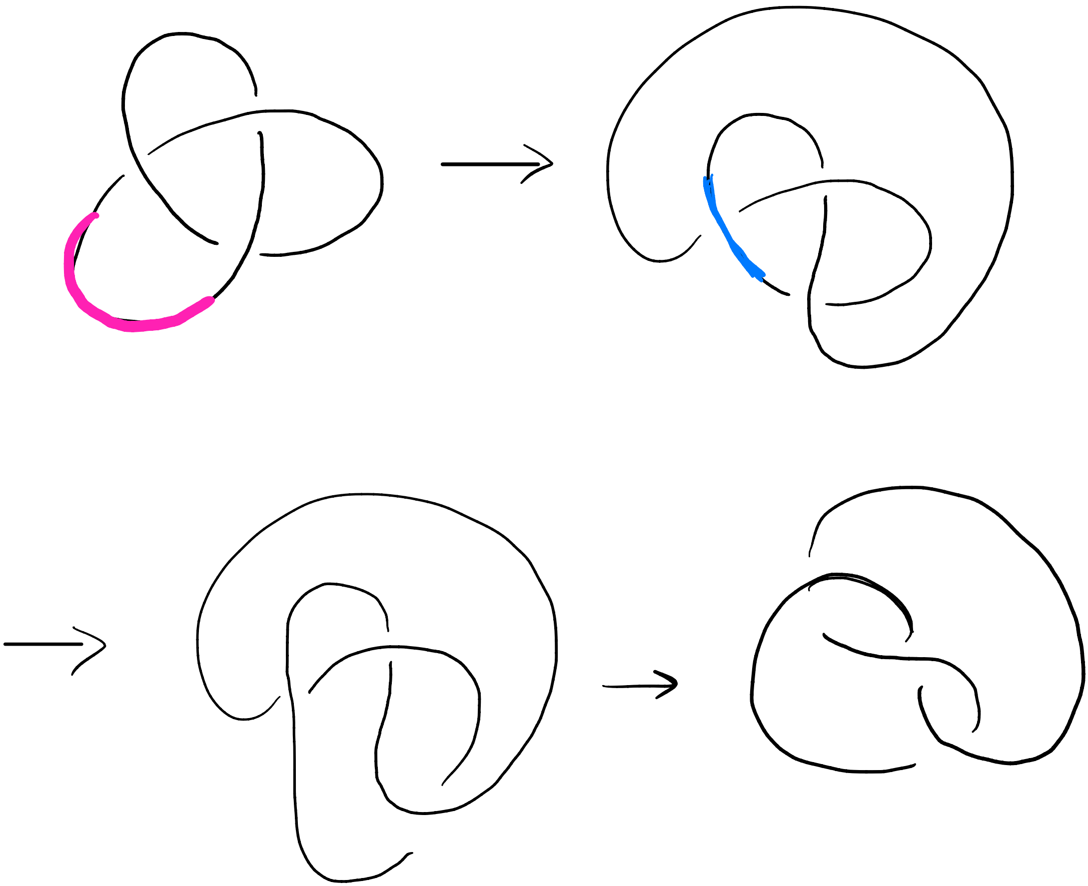


slide

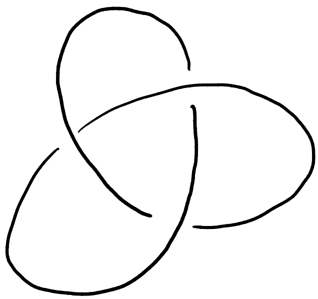


Ex:

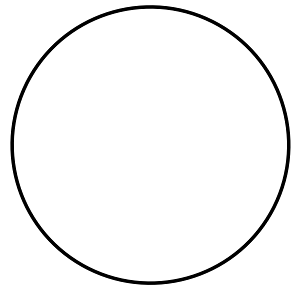




Remark:

 $\not\simeq$ isotopic

unknot



Need Knot Invariants.

Why study Knots?

1. Easy to state questions about knots that are really hard to answer.

2. Use knots to construct 3 and 4-mans.

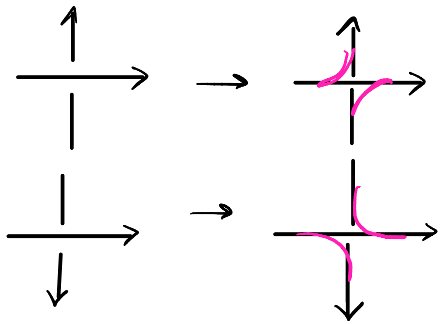
Spanning Surfaces / Seifert Surfaces

Spanning Surface of a Knot K:

A surface whose boundary is K.

Seifert Algorithm

1. Start with a projection of K.
2. Give it an orientation
3. Eliminate the crossings:
   - At each crossing two strands come in and two come out
   - Connect each of the strands coming into the crossing to the adjacent strand leaving the crossing.



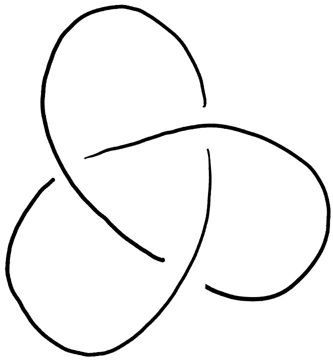
4. Fill in the circles.
(each circle will bound a disk

5. Color the disks

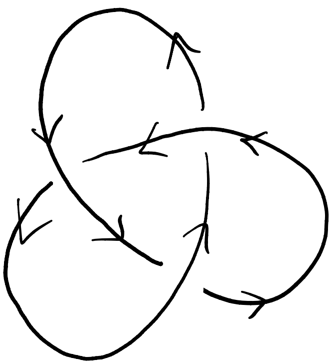
6. Connect the disks to one another,
at the crossings of the knots,
by twisted bands.

Ex:

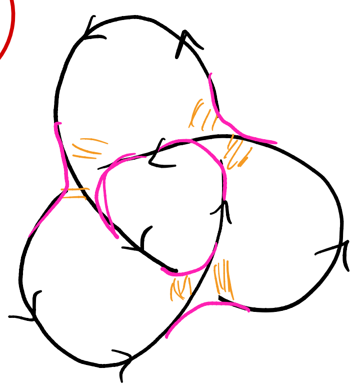
1

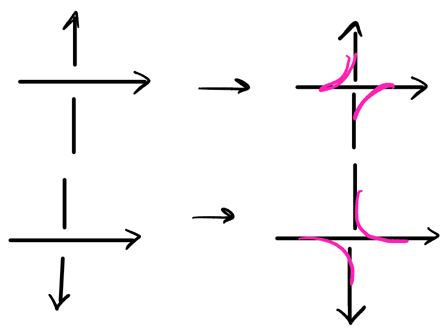


2

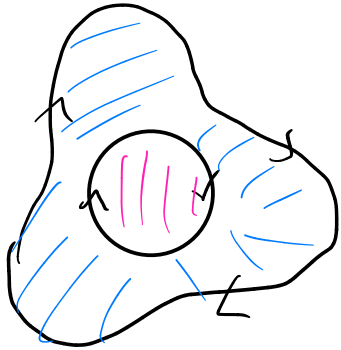


3

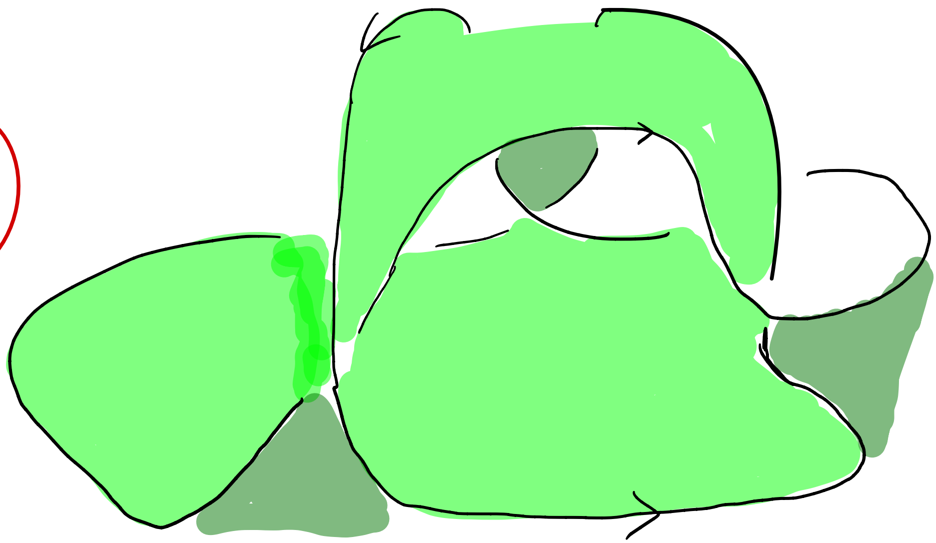




4 + 5



6



This is a compact orientable spanning surface for K.

bands = 3

disks = 2

bdry components = 1

$$\Rightarrow \chi(S) = 2 - 3 + 1 = 0$$

$$\Rightarrow g = 1$$

may NOT be the genus
of the link/knot.

Defn:

$g(L) = \min\{\text{genus of } S :$

S is a Seifert Surface for L

Remarks:

1. Seifert Algorithim may or may NOT produce minimal genus (max χ) surface.

So, finding the minimal genus can be difficult.

(2) Minimal Seifert Surface is NOT unique.

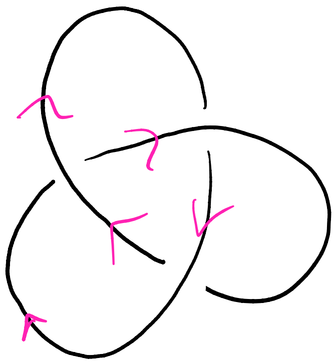
i.e. a given knot may have two different Seifert surfaces of least genus.

(3) $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Remarks

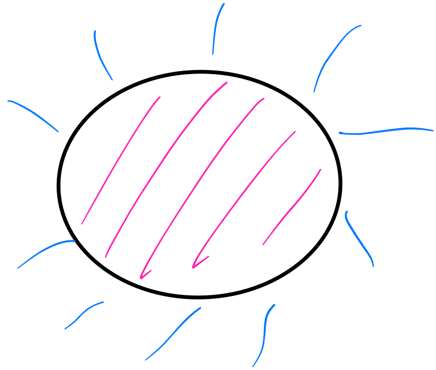
① These surfaces may OR may NOT be orientable.

Exercise:



② A surface S is non-orientable $\Longleftrightarrow$ $MB \subset S$

Example:

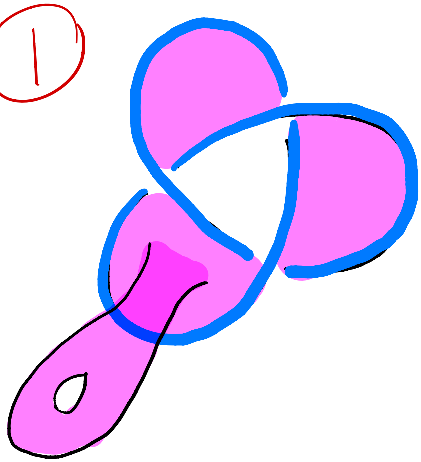


unknot bounds a disk,

disk spanning surface for the unknot.

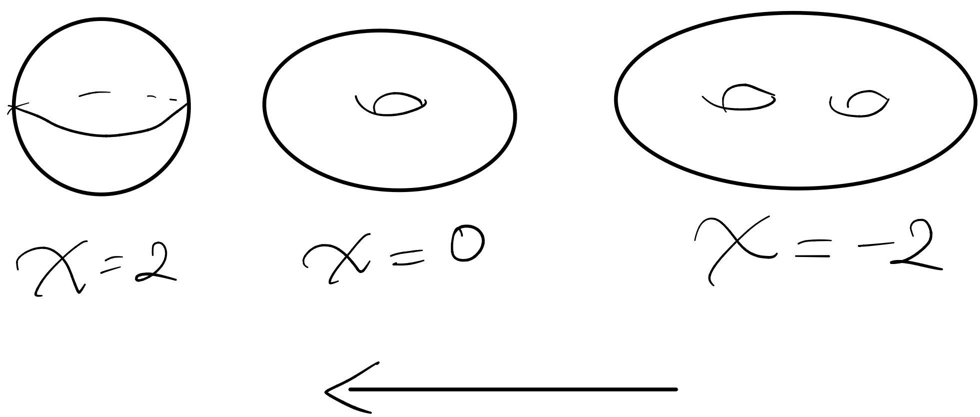
(only knot with this property up to isotopy)

Remarks:

1  is also a spanning surface for trefoil knot but a wierd one.

Usefull to consider "minimal" spanning surfaces.

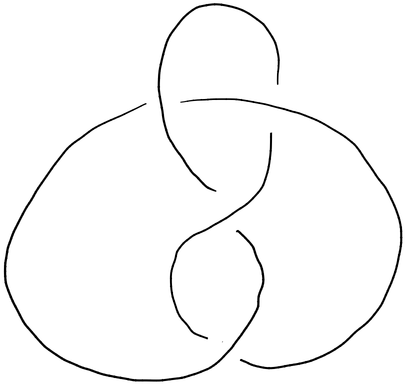
i.e. max χ.



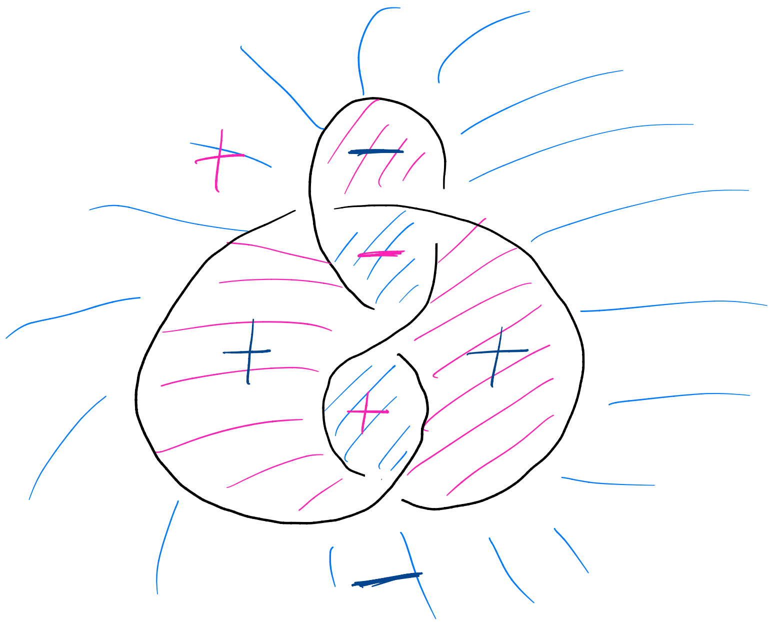
② We don't want to consider spanning surfaces with closed components

③ Sometimes want to restrict to orientable surfaces.

Ex: Figure 8 knot 4_1



Q: Which of these orientable?



NEITHER

Links:

$$L \underset{\text{embedding}}{\hookrightarrow} M^3$$

Embeding of $\bigcup_{i=1}^{n} S^1 \hookrightarrow M$

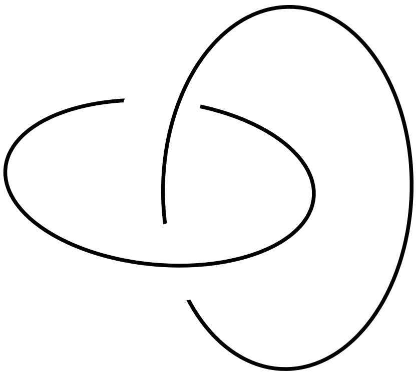
OR

1-dim. submanifold of M^3

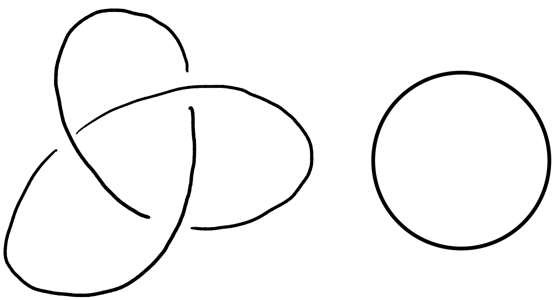
with NO boundary.

Examples:

① Hopf Link:

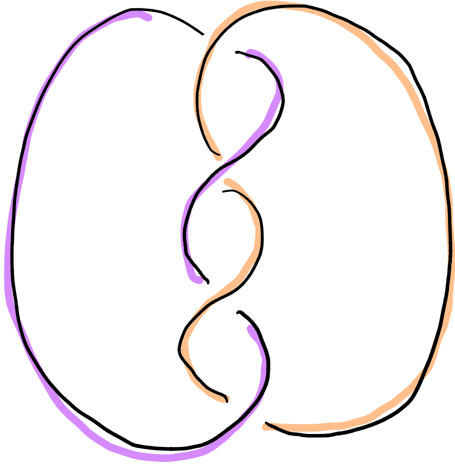


2-components

② 

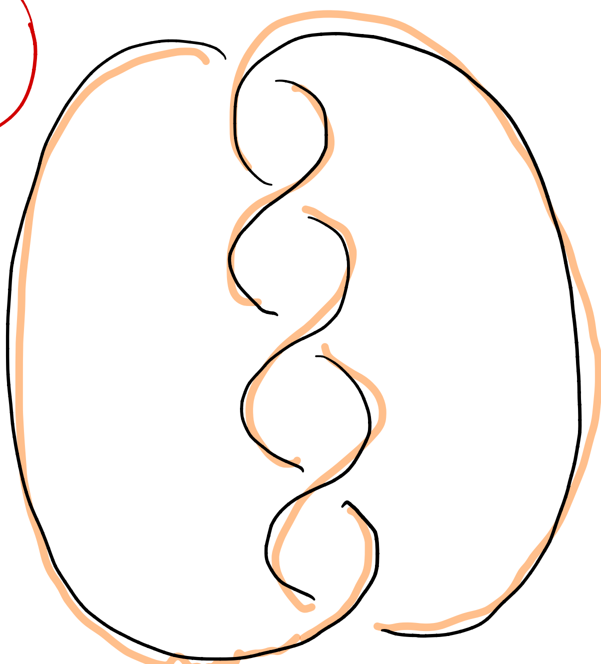
2-components

③



2-components

④



Knot

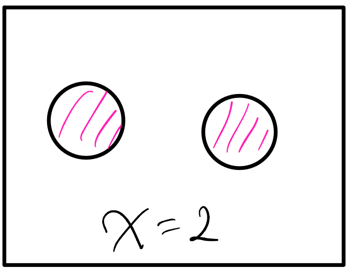
1-component (not a link)

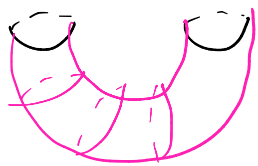
Remark:

- Diagrams make sense.
- Seifert Algorithm

Q: What is different?

① Links can bound disconnected surfaces.



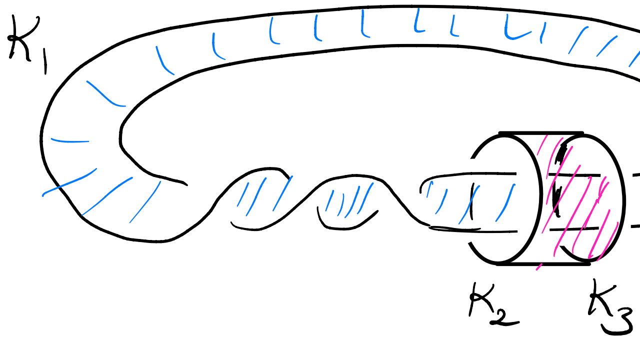


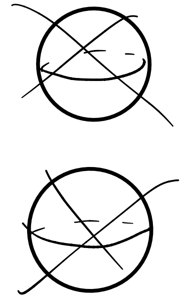
$\chi = 0$

② even restrict to orientable surfaces

"minimal genus"

max χ (don't allow closed components.)

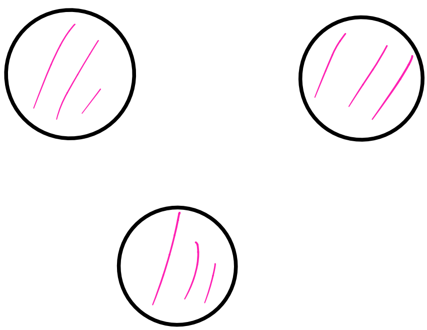


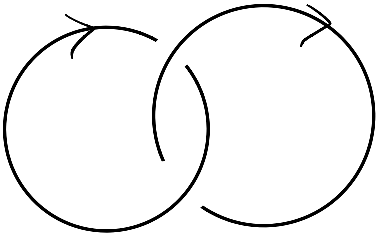
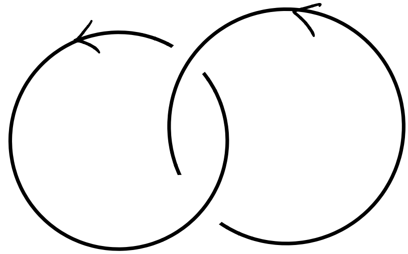


$$L = K_1 \cup K_2 \cup K_3$$

Q: what is the max χ spanning surface this link could have?

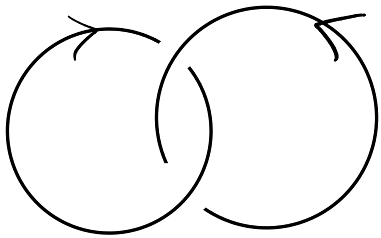
Any 3-component link?



 $\cong$ 

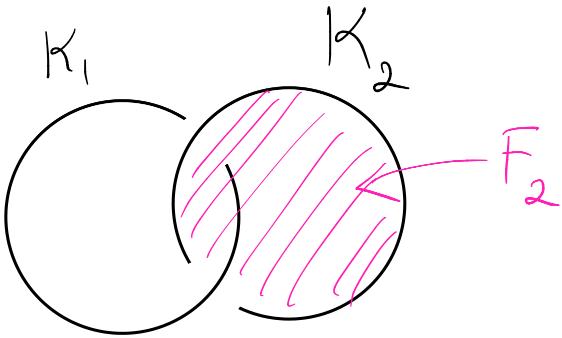
isotopic

as oriented links

Remark: 

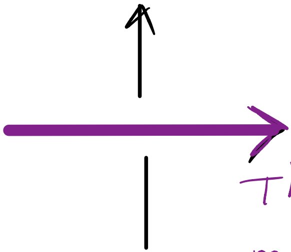
not isotopic to the previous two.

Linking Number



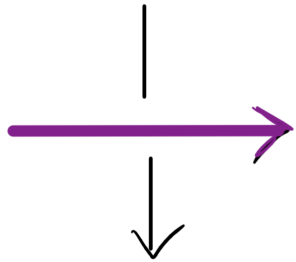
$lk(K_1, K_2) = \underset{\text{oriented count}}{\#} K_1 \cap F_2$

Linking number between two components of an (oriented) link, say K_1 and K_2, is an oriented count of intersections between K_1 and F_2



Thumb of my right hand

$+1$



-1

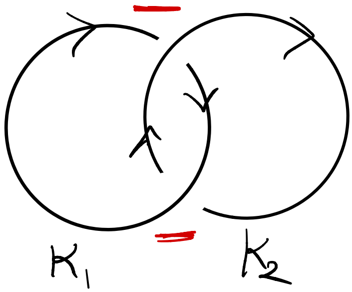
$$\boxed{lk(K_1, K_2) := \frac{\#(+1) + \#(-1)}{2}}$$

Remark: lk doesn't depend on the choice of the Seifert Surface.

$$lk(K_1, K_2) = \underset{\text{oriented count}}{\#} F_1 \cap K_2$$

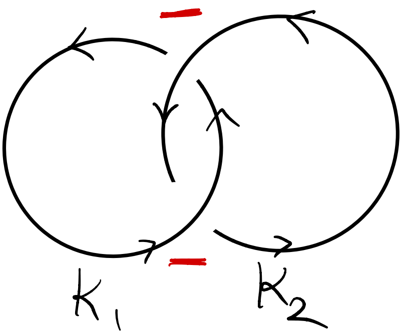
Examples:

1



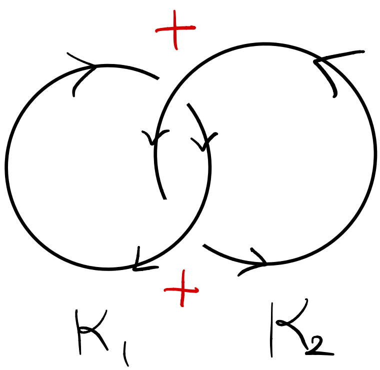
$$lk(K_1,K_2) = \frac{-2}{2} = -1$$

2



$$lk(K_1,K_2) = \frac{-2}{2} = -1$$

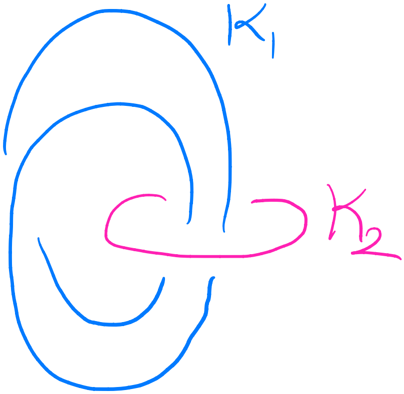
3

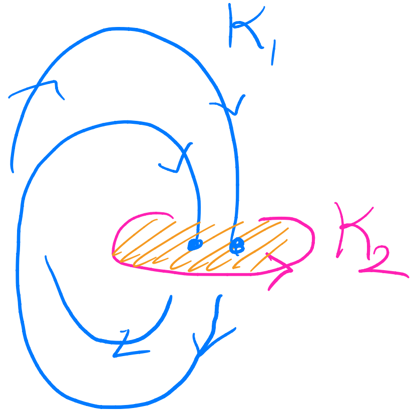


$$lk(K_1,K_2) = \frac{2}{2} = 1$$

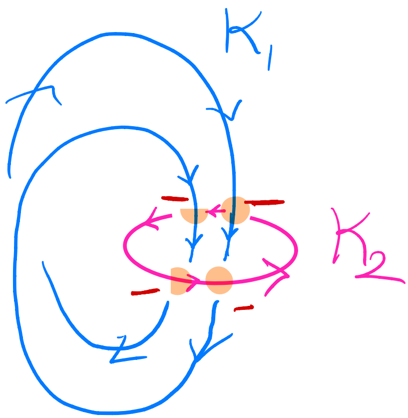
$\Rightarrow$ 1, 2 $\not\simeq$ 3

4





$$lk(K_1, K_2) = \pm 2$$



$$\Rightarrow lk(K_1, K_2) = \frac{-4}{2} = -2$$

Facts about linking number

① Invariant of oriented links

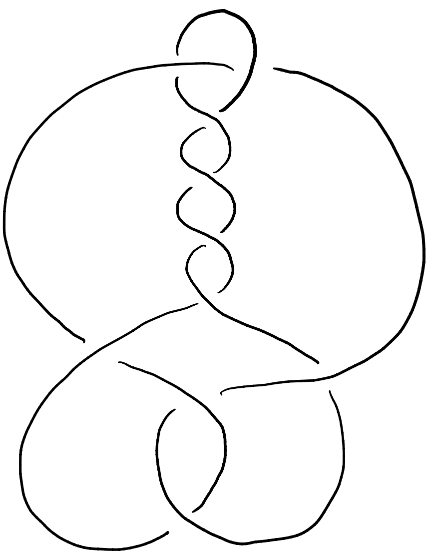
② Symmetric:

$$lk(K_1, K_2) = lk(K_2, K_1)$$

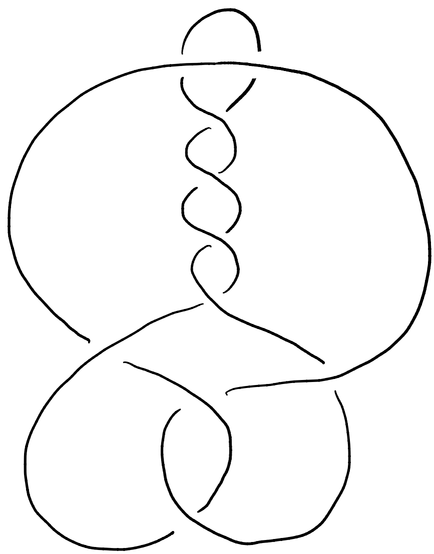
Alternating Knots

A knot with a projection that has crossing that alternates between over and under.

Example

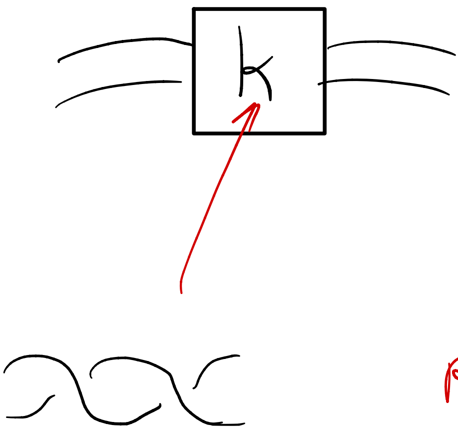


alternating

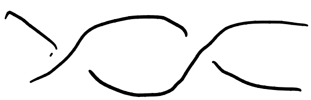


not alternating

Q: How can I describe an infinite family of knots?



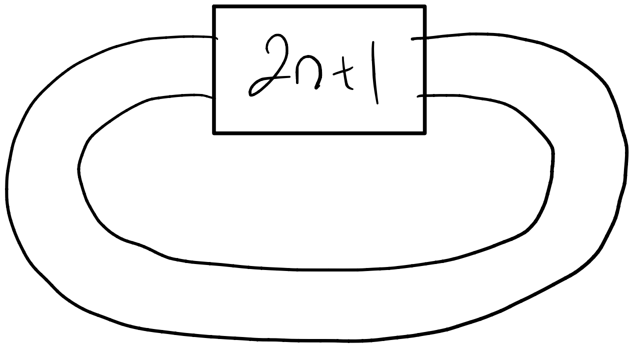
RH (Right-Handed)



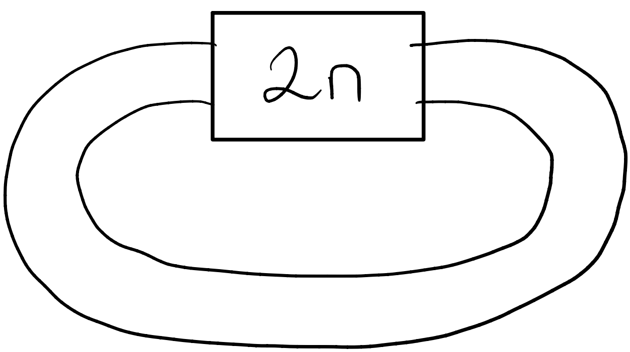
LH (Left Handed)

Examples:

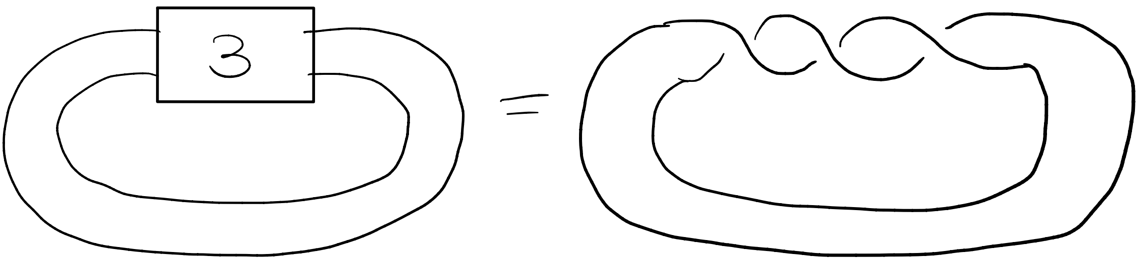
① $T(2, 2n+1)$ Torus knots



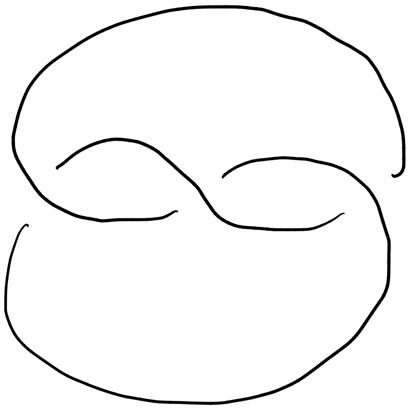
② $T(2, 2n)$ 2-component link



Ex: $T(2,3) = ?$



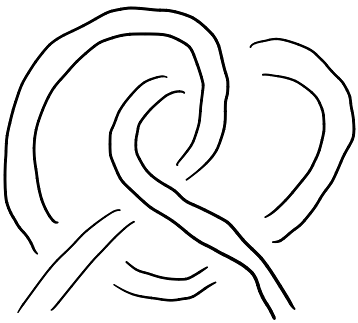
$\rightarrow$

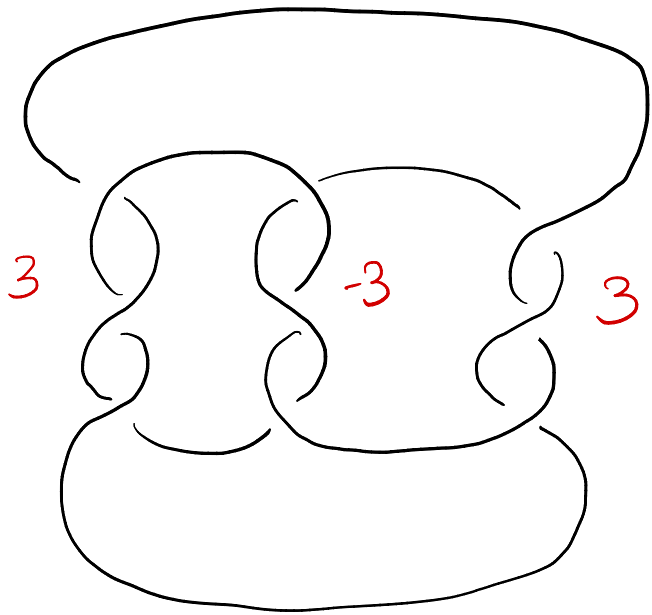


trefoil knot

3_1

3 Pretzel Links

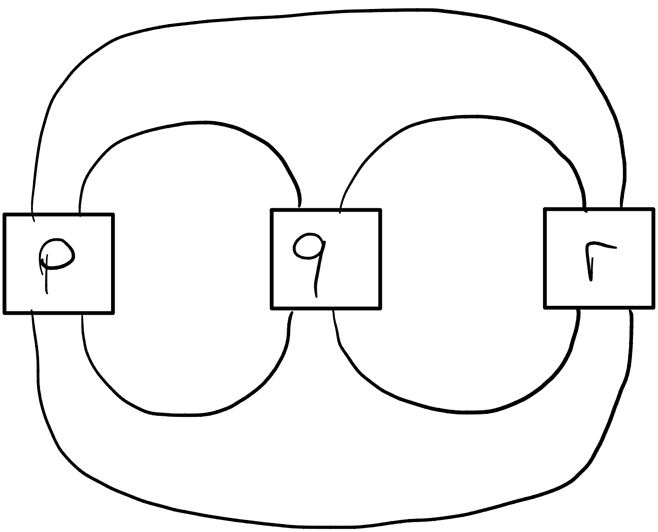




9_{46}

$P(3, -3, 3)$

3-strand Pretzel Link $P(p,q,r)$



4-strand Pretzel Links $P(p,q,r,s)$