Donaldson's Obstruction

Def: A rational homology 3-sphere \((\mathbb{Q}S^3)\) is a closed 3-manifold \(Y\) such that

\[H_1(Y) \text{ and } H_2(Y) \text{ are finite groups}\]

\[(H_0(Y) = H_3(Y) \cong \mathbb{Z} \text{ by default})\]

Equivalently, \(H_1(Y; \mathbb{Q}) = H_2(Y; \mathbb{Q}) = 0\)

(i.e. the rational homology of \(Y = \text{rational homology of } S^3\))

A rational homology 4-ball \((\mathbb{Q}B^4)\) is a 4-manifold \(X\) with boundary such that \(H_1(X), H_2(X), H_3(X)\) are finite groups

\[(H_0(X) \cong \mathbb{Z}, H_4(X) = 0 \text{ by default})\]

Fact: If \(X\) is a \(\mathbb{Q}B^4\), then \(\partial X\) is a \(\mathbb{Q}S^3\)

But: Not all \(\mathbb{Q}S^3\)s bound a \(\mathbb{Q}B^4\).

Ex: Poincaré Homology Sphere

Let \(X = \begin{array}{cccccc}
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}\), \(Y = \partial X\)

The boundary of \(X\) is called the Poincaré homology sphere.

Notice, \(Q_x = \begin{bmatrix}
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
\end{bmatrix}\), \(\det Q_x = 1\), \(Q_x\) is positive-definite

Recall, \(Q_x = E_8\)
Assume $Y$ bounds a $QB^4$, $B$.
Then $Z = X \cup (-B)$ is a closed positive definite 4-mfld

By Donaldson's theorem, $Qz$ is diagonalizable and so $I$ basis s.t. $Q_z = I$

i.e. $I$ lattice isomorphism $(\mathbb{Z}^8, Q_z) \cong (\mathbb{Z}^8, I)$

Since $X \subset Z$, $(\mathbb{Z}^8, Q_x)$ can be viewed as a sublattice of $(\mathbb{Z}^8, Q_z) \cong (\mathbb{Z}^8, I)$

i.e. $I$ lattice embedding $(\mathbb{Z}^9, Q_x) \hookrightarrow (\mathbb{Z}^8, I)$

Since $\det Q_x = 1 = \det Q_z$, this is an isomorphism

$\Rightarrow I$ lattice isomorphism $(\mathbb{Z}^8, E_8) \rightarrow (\mathbb{Z}^8, I)$

But by previous homework, no such embedding exists.

Thus $Y$ does not bound a $QB^4$. 
More generally:

Suppose $Y$ is a 3-mfd that bounds a pos/neg-definite 4-mfld $X$ and a rational homology 4-ball $B$. Then $Z = X \cup_Y (-B)$ is a closed pos/neg-definite 4-mfd with $\text{rank } H_2(X) = \text{rank } H_2(Z) = n$

By Donaldson, we have an isomorphism $(Z^n, Q_Z) \cong (Z^n, \pm I)$

Choose a basis for $H_2(X)$ and let $Q_x$ be its intersection matrix.

Then there is a lattice embedding $(H_2(P), Q_0) \hookrightarrow (H_2(X), \pm I_n)$

(If $\det Q_x = 1$, then this is an isomorphism; equivalent to saying $Q_x$ is diagonalizable, as in $E_8$ example.)

Donaldson's Obstruction

If $Y$ bounds a 3B4 and $X$ is a pos/neg-def 4-mfd with $\partial X = Y$, then there is a lattice embedding $\varphi: (Z^n, Q_x) \rightarrow (Z^n, \pm I)$
Ex: **Lens Spaces**

Let \( X = \mathbb{S}_{a_1, a_2, \ldots, a_n} \) \( a_i \geq 2 \ \forall i \) (neg-def)

Set \( \mathbb{Q} = [a_1, \ldots, a_n] = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{a_4 - \ldots}}} \)

Then \( \partial X \) is called a **Lens Space**

and it is denoted by \( L(p, q) \)

Lens Spaces are \( \mathbb{C}S^3 \)'s.

Q: Which lens spaces bound \( \mathbb{Q}B^4 \)'s?

This was answered by Lisca in '07. (paper on site)

He gave a list of 7 infinitely families of lens spaces that bound \( \mathbb{Q}B^4 \)'s

To show all others do not bound \( \mathbb{Q}B^4 \)'s, he showed

\( \mathbb{Z} \) lattice embedding \( (\mathbb{Z}^n, \mathbb{Q}_x) \rightarrow (\mathbb{Z}^n, -I) \)

(a lot of work)

Read "Lens spaces, rational balls, and the ribbon conjecture" by Lisca starting w/ Section 2.
Double Branched Covers

Given a link $L \subset S^3$, we can form what is called the double cover of $S^3$ branched over $L$, which we denote by $\Sigma_2(L)$.

This is a 3-manifold and often, it is a $QS^3$.

Given a spanning surface $F \subset B^4$ for $L$, we can also form the double cover of $B^4$ branched over $F$, which we denote by $\Sigma_2(F)$.

This is a 4-manifold.

**Thm (Donald-Owens):** If $L$ is $\times$-slice and $\det L \neq 0$, then $\Sigma_2(L)$ bounds a $QS^4$ namely $\Sigma_2(F)$, where $F \subset B^4$ is a surface with $\chi(F) = 1$.

Obstruction to $\times$-sliceness

If $L$ is $\times$-slice with $\det L \neq 0$ and $\Sigma_2(L)$ bounds a definite 4-manifold $X$,

then 3 lattice embedding $(H_2(X), \mathbb{Q}_X) \to \mathbb{Z}^{\text{rank}(H_2(X))} \pm 1$. 

Ex: 3-stranded pretzel knots

\[
\text{let } L = \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\]

If \( p,q > 0, q < 0 \), then

\[
\Sigma_2(L) = \begin{pmatrix}
2 & -2 & 2 & -2 \\
-2 & 2 & -2 & 2 \\
2 & -2 & 2 & -2 \\
\vdots & \vdots & \vdots & \vdots \\
2 & -2 & 2 & -2 \\
\end{pmatrix}
\]

Fact: negdef if \( \frac{1}{p} + \frac{1}{2} + \frac{1}{r} > 0 \)

\[
\text{or }
\]

Let \( Q_x \) be the intersection form. Then if \( L \) is \( \lambda \)-slice,

\[\] lattice embedding \( (\mathbb{Z}^{p+q}, Q_x) \to (\mathbb{Z}^{p+q}, -I) \)

\( \text{(when } \frac{1}{p} + \frac{1}{2} + \frac{1}{r} > 0) \)