Donaldson's Obstruction

Def: A rational homology 3-sphere ($\mathbb{Q}S^3$) is a closed 3-manifold Y such that

$$H_1(Y) \text{ and } H_2(Y) \text{ are finite groups}$$
$$(H_0(Y) = H_3(Y) \cong \mathbb{Z} \text{ by default})$$

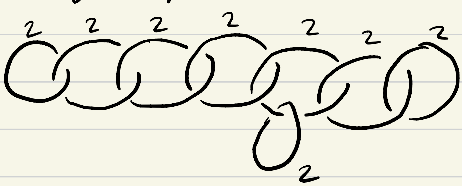
Equivalently, $H_1(Y;\mathbb{Q}) = H_2(Y;\mathbb{Q}) = 0$
(i.e. the rational homology of Y = rational homology of S^3)

A rational homology 4-ball ($\mathbb{Q}B^4$) is a 4-manifold X with boundary such that $H_1(X), H_2(X), H_3(X)$ are finite groups
$(H_0(X) \cong \mathbb{Z},\ H_4(X) = 0$ by default$)$

Fact: If X is a $\mathbb{Q}B^4$, then ∂X is a $\mathbb{Q}S^3$

But: Not all $\mathbb{Q}S^3$s bound a $\mathbb{Q}B^4$.

Ex: Poincaré Homology Sphere

Let $X =$ 

, $Y = \partial X$

The boundary of X is called the Poincaré homology sphere

Notice, $Q_X = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$, $\det Q_X = 1$, Q_X is positive-definite

Recall, $Q_X = E_8$

Assume Y bounds a $\mathbb{Q}B^4$, B.

Then $Z = X \cup (-B)$ is a closed positive definite 4-mfld

By Donaldson's theorem, Q_Z is diagonalizable and so $\exists$ basis s.t. $Q_Z = I$

i.e. $\exists$ lattice isomorphism $(\mathbb{Z}^8, Q_Z) \cong (\mathbb{Z}^8, I)$

Since $X \subset Z$, $(\mathbb{Z}^8, Q_X)$ can be viewed as a sublattice of $(\mathbb{Z}^8, Q_Z) \cong (\mathbb{Z}^8, I)$

i.e. $\exists$ lattice embedding $(\mathbb{Z}^8, Q_X) \longrightarrow (\mathbb{Z}^8, I)$

Since $\det Q_X = 1 = \det Q_Z$, this is an isomorphism

$\Rightarrow \exists$ lattice isomorphism $(\mathbb{Z}^8, E_8) \rightarrow (\mathbb{Z}^8, I)$

But by previous homework, no such embedding exists.

Thus Y does not bound a $\mathbb{Q}B^4$.

More generally:

Suppose Y is a 3-mfld that bounds a pos/neg-definite 4-mfld X and a rational homology 4-ball B.
Then $Z = X \cup_Y (-B)$ is a closed pos/neg-definite 4-manifold with $\operatorname{rank} H_2(X) = \operatorname{rank} H_2(Z) = n$

By Donaldson, we $\exists$ lattice isom. $(\mathbb{Z}^n, Q_Z) \cong (\mathbb{Z}^n, \pm I)$

Choose a basis for $H_2(X)$ and let Q_X be its intersection matrix.

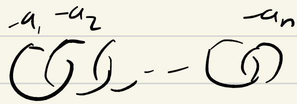
Then $\exists$ a lattice embedding $(H_2(P), Q_X) \hookrightarrow (H_2(X), \pm I_n)$

(If $|\det Q_X| = 1$, then this is an isom & equivalent to saying Q_X is diagonalizable, as in E_8 example)

Donaldson's Obstruction

If Y bounds a $\mathbb{Q}B^4$ & X is a pos/neg-def 4-mfld w/ $\partial X = Y$, then $\exists$ lattice embedding $\varphi: (\mathbb{Z}^n, Q_X) \to (\mathbb{Z}^n, \pm I)$

Ex: Lens Spaces

Let $X =$ 

$a_i \geq 2 \ \forall i$ (neg-def)

Set $\frac{p}{q} = [a_1, \ldots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$

Then ∂X is called a Lens Space

and it is denoted by $L(p,q)$

Lens Spaces are $\mathbb{Q}S^3$'s.

Q: Which Lens spaces bound $\mathbb{Q}B^4$'s?

This was answered by Lisca in '07. (paper on site)

He gave a list of 7 infinitely families of Lens spaces that bound $\mathbb{Q}B^4$'s

To show all others do not bound $\mathbb{Q}B^4$'s, he showed

$$\nexists \text{ lattice embedding } (\mathbb{Z}^n, Q_X) \longrightarrow (\mathbb{Z}^n, -I)$$

(a lot of work)

Read "Lens spaces, rational balls, and the ribbon conjecture" by Lisca starting w/ Section 2.

Double Branched Covers

Given a link $L \subset S^3$, we can form what is called the double cover of S^3 branched over L, which we denote by $\Sigma_2(L)$

This is a 3-manifold and often, it is a $\mathbb{Q}S^3$.

Given a spanning surface $F \subset B^4$ for L, we can also form the double cover of B^4 branched over F, which we denote by $\Sigma_2(F)$

This is a 4-manifold.

will define later

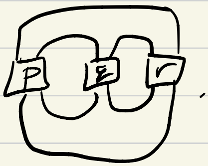
Thm (Donald-Owens): If L is χ-slice and $\det L \neq 0$, then $\Sigma_2(L)$ bounds a $\mathbb{Q}B^4$, namely $\Sigma_2(F)$, where $F \subset B^4$ is a surface w/ $\chi(F) = 1$.

Obstruction to χ-sliceness

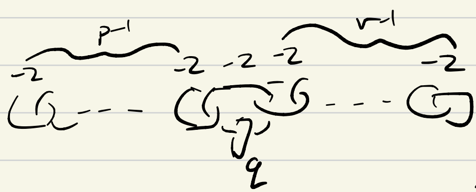
If L is χ-slice w/ $\det L \neq 0$ and $\Sigma_2(L)$ bounds a definite 4-mfld X,

then $\exists$ lattice embedding $(H_2(X), Q_X) \to (\mathbb{Z}^{\operatorname{rank}(H_2(X))}, \pm I)$

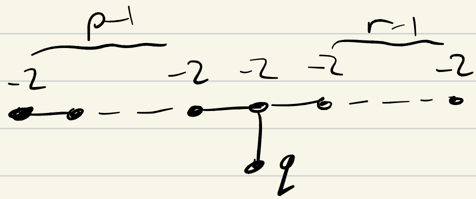
Ex: 3-stranded pretzel knots

Let $L =$  .

If $p, q > 0$, $q < 0$, then

$\Sigma_2(L) = \partial \Big($  $\Big)$

Fact: neg def if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 0$

OR 

Let Q_X be the intersection form.
Then if L is χ-slice,

$\exists$ lattice embedding $(\mathbb{Z}^{p+r}, Q_X) \to (\mathbb{Z}^{p+r}, -I)$

(when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 0$)